Potential Wadge classes

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Abstract. Let $\mathbf{\Gamma}$ be a Borel class, or a Wadge class of Borel sets, and $2 \leq d \leq \omega$ a cardinal. We study the Borel subsets of $\mathbb{R}^d$ that can be made $\mathbf{\Gamma}$ by refining the Polish topology on the real line. These sets are called potentially $\mathbf{\Gamma}$. We give a test to recognize potentially $\mathbf{\Gamma}$ sets.

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1 Introduction

The reader should see [K] for the descriptive set theoretic notation used in this paper. The standard way of comparing the topological complexity of subsets of 0-dimensional Polish spaces is the Wadge reducibility quasi-order $\leq_W$. Recall that if X (resp., Y) is a 0-dimensional Polish space and A (resp., B) a subset of X (resp., Y), then

$$(X,A)\leq_W (Y,B)\ \Leftrightarrow\ \exists f\!:\!X\!\rightarrow\!Y \text{ continuous such that } A\!=\!f^{-1}(B).$$

This is a very natural definition since the continuous functions are the morphisms for the topological structure. So the scheme is as follows:

$$\begin{array}{c|c|c|c|c} \cline{2-2}\cline{4-4} X & A & \dashrightarrow & B & Y \\ \cline{2-2}\cline{4-4} & \neg A & \dashrightarrow & \neg B & \\ \cline{2-2}\cline{4-4} \end{array}$$

The "0-dimensional" condition is here to ensure the existence of enough continuous functions (the only continuous functions from $\mathbb{R}$ into ω^ω are the constant functions, for example). In the sequel, $\mathbf{\Gamma}$ will be a class of Borel subsets of 0-dimensional Polish spaces. We denote by $\check{\mathbf{\Gamma}}:=\{\neg A\mid A\in\mathbf{\Gamma}\}$ the class of complements of elements of $\mathbf{\Gamma}$. We say that $\mathbf{\Gamma}$ is $self\text{-}dual$ if $\mathbf{\Gamma}=\check{\mathbf{\Gamma}}$. We also set $\Delta(\mathbf{\Gamma}):=\mathbf{\Gamma}\cap\check{\mathbf{\Gamma}}$. Following 4.1 in [Lo-SR2], we give the following definition:

Definition 1.1 *We say that $\mathbf{\Gamma}$ is a $Wadge\ class\ of\ Borel\ sets$ if there is a Borel subset A_0 of ω^ω such that for each 0-dimensional Polish space X, and for each $A\subseteq X$, A is in $\mathbf{\Gamma}$ if and only if $(X,A)\leq_W(\omega^\omega,A_0)$. We say that A_0 is $\mathbf{\Gamma}$-$complete$.*

The Wadge hierarchy defined by $\leq_W$, i.e., the inclusion of Wadge classes, is the finest hierarchy of topological complexity in descriptive set theory. The goal of this paper is to study the descriptive complexity of the Borel subsets of products of Polish spaces. More specifically, we are looking for a dichotomy of the following form, quite standard in descriptive set theory: either a set is simple, or it is more complicated than a well-known complicated set. Of course, we have to specify the notions of complexity and comparison we are considering. The two things are actually very much related. The usual notion of comparison between analytic equivalence relations is the Borel reducibility quasi-order $\leq_B$. Recall that if X (resp., Y) is a Polish space and E (resp., F) an equivalence relation on X (resp., Y), then $(X,E)\leq_B(Y,F)\ \Leftrightarrow\ \exists f:X\rightarrow Y$ Borel such that $E=(f\times f)^{-1}(F)$. Note that this makes sense even if E and F are not equivalence relations. The notion of complexity we are considering is a natural invariant for $\leq_B$ in dimension 2. Its definition generalizes Definition 3.3 in [Lo3] to any dimension d making sense in the context of descriptive set theory, and also to any class $\mathbf{\Gamma}$. So in the sequel d will be a cardinal, and we will have $2\!\leq\!d\!\leq\!\omega$ since 2^{ω_1} is not metrizable.

Definition 1.2 *Let $(X_i)_{i\in d}$ be a sequence of Polish spaces, and B a Borel subset of $\Pi_{i\in d}\ X_i$. We say that B is $potentially\ in$ $\mathbf{\Gamma}$ $\big($denoted $B\!\in\!pot(\mathbf{\Gamma})\big)$ if, for each $i\!\in\!d$, there is a finer 0-dimensional Polish topology τ_i on X_i such that $B\!\in\!\mathbf{\Gamma}\big(\Pi_{i\in d}\ (X_i,\tau_i)\big)$.*

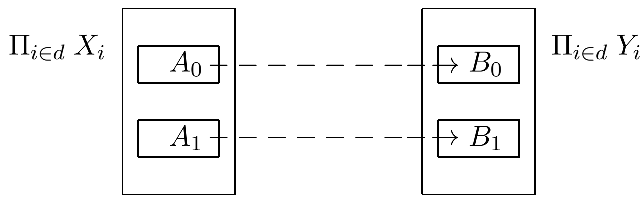
One should emphasize the fact that the point of this definition is to consider product topologies. Indeed, if B is a Borel subset of a Polish space X, then there is a finer Polish topology τ on X such that B is a clopen subset of (X,τ) (see 13.1 in [K]). This is not the case in products: if for example $\mathbf{\Gamma}$ is a non self-dual Wadge class of Borel sets, then there are sets in $\mathbf{\Gamma}\big((\omega^\omega)^2\big)$ that are not pot$(\check{\mathbf{\Gamma}})$ (see Theorem 3.3 in [L1]). For example, the diagonal of ω^ω is not potentially open.

Note also that since we work up to finer Polish topologies, the "0-dimensional" condition is not a restriction. Indeed, if X is a Polish space, then there is a finer 0-dimensional Polish topology on X (see 13.5 in [K]). The notion of potential complexity is an invariant for $\leq_B$ in the sense that if $(X,E)\leq_B (Y,F)$ and F is pot$(\mathbf{\Gamma})$, then E is pot$(\mathbf{\Gamma})$ too.

The good notion of comparison is not the rectangular version of $\leq_B$. Instead of considering a Borel set E and its complement, we have to consider pairs of disjoint analytic sets. This leads to the following notation. Let $(X_i)_{i\in d}$, $(Y_i)_{i\in d}$ be sequences of Polish spaces, and A_0, A_1 (resp., B_0, B_1) disjoint analytic subsets of $\Pi_{i\in d}\ X_i$ (resp., $\Pi_{i\in d}\ Y_i$). Then

$$\big((X_i)_{i\in d},A_0,A_1\big)\leq\big((Y_i)_{i\in d},B_0,B_1\big)\ \Leftrightarrow\ \forall i\!\in\! d\ \exists f_i\!:\!X_i\!\rightarrow\! Y_i \text{ continuous such that}$$
$$\forall\varepsilon\!\in\! 2\ \ A_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(B_\varepsilon).$$

So the good scheme of comparison is as follows:



The notion of potential complexity was studied in [L1]-[L7] for $d=2$ and the non self-dual Borel classes. The main question of this long study was asked by A. Louveau to the author in 1990. A. Louveau wanted to know whether Hurewicz's characterization of G_δ sets could be extended to pot$(\mathbf{\Gamma})$ sets when $\mathbf{\Gamma}$ is a Wadge class of Borel sets. The main result of this paper gives a complete and positive answer to this question:

Theorem 1.3 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, or the class $\mathbf{\Delta}^0_\xi$ for some $1\!\leq\!\xi\!<\!\omega_1$. Then there are Borel subsets $\mathbb{S}^0$, $\mathbb{S}^1$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*
(a) The set A_0 is separable from A_1 by a pot$(\mathbf{\Gamma})$ set.
(b) The inequality $\big((d^\omega)_{i\in d},\mathbb{S}^0,\mathbb{S}^1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

It is natural to try to prove Theorem 1.3 since it is a result of continuous reduction, which appears in the very definition of a Wadge class. So it goes beyond a simple generalization. The work in this paper is the continuation of the article [L7], that was announced in [L6]. We generalize the main results of [L7]. The generalization goes in different directions:
- It works in any dimension d.
- It works for the self-dual Borel classes $\mathbf{\Delta}^0_\xi$.
- It works for any Wadge class of Borel sets, which is the hardest part.

We generalize, and also in fact give a new proof of the dimension 1 version of Theorem 1.3 obtained by A. Louveau and J. Saint Raymond (see [Lo-SR1]), which itself was a generalization of Hurewicz's result. The new proof is without games, and gives a new approach to the study of Wadge classes. Note that A. Louveau and J. Saint Raymond proved that if $\mathbf{\Gamma}$ is not self-dual, then the reduction map in (b) can be one-to-one (see Theorem 5.2 in [Lo-SR2]). We will see that there is no injectivity in general in Theorem 1.3. However, G. Debs proved that we can have the f_i's one-to-one when $d\!=\!2$, $\mathbf{\Gamma}\!\in\!\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi\}$ and $\xi\!\geq\!3$. Some injectivity details will be given in the last section.

We introduce the following notation and definition in order to specify Theorem 1.3. One can prove that a reduction on the whole product is not possible, for acyclicity reasons (see [L5]-[L7]). We now specify this. We emphasize the fact that in this paper, there will be a constant identification between $(d^d)^l$ and $(d^l)^d$, for $l\leq\omega$, to avoid as much as possible heavy notation.

Notation. If $\mathcal{X}$ is a set, then $\vec{x}:=(x_i)_{i\in d}$ is an arbitrary element of $\mathcal{X}^d$. If $\mathcal{T}\subseteq\mathcal{X}^d$, then we denote by $G^{\mathcal{T}}$ the graph with set of vertices $\mathcal{T}$, and with set of edges $\big\{\{\vec{x},\vec{y}\}\subseteq\mathcal{T}\mid \vec{x}\neq\vec{y}\ \text{ and }\ \exists i\in d\ \ x_i=y_i\big\}$ (see [B] for the basic notions about graphs). So $\vec{x}\neq\vec{y}\in\mathcal{T}$ are $G^{\mathcal{T}}$-related if they have at least one coordinate in common.

Definition 1.4 *(a) We say that $\mathcal{T}$ is $one\text{-}sided$ if the following holds:*

$$\forall\vec{x}\neq\vec{y}\in\mathcal{T}\ \ \forall i\neq j\in d\ \ (x_i\neq y_i\vee x_j\neq y_j).$$

This means that if $\vec{x}\neq\vec{y}\in\mathcal{T}$, then they have at most one coordinate in common.

(b) We say that $\mathcal{T}$ is $almost\ acyclic$ if for every $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n\leq L}$ there are $i\in d$ and $k<m<n<L$ such that $x^k_i=x^m_i=x^n_i$. This means that every $G^{\mathcal{T}}$-cycle contains a "flat" subcycle, i.e., a subcycle in a single direction $i\in d$.

(c) We say that a tree T on d^d is a $tree\ with\ suitable\ levels$ if the set $\mathcal{T}^l:=T\cap(d^d)^l\subseteq(d^l)^d$ is finite, one-sided and almost acyclic for each integer l.

We do not really need the finiteness of the levels, but it makes the proof of Theorem 1.3 much simpler. The following classical property will be crucial in the sequel:

Definition 1.5 *We say that $\mathbf{\Gamma}$ has the $separation\ property$ if for each $A,B\in\mathbf{\Gamma}(\omega^\omega)$ disjoint, there is $C\in\Delta(\mathbf{\Gamma})(\omega^\omega)$ separating A from B.*

The separation property has been studied in [S] and [vW], where the following is proved:

Theorem 1.6 *(Steel-van Wesep) Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets. Then exactly one of the two classes $\mathbf{\Gamma}$, $\check{\mathbf{\Gamma}}$ has the separation property.*

We now specify Theorem 1.3.

Theorem 1.7 *We can find a tree T_d with suitable levels, together with, for each non self-dual Wadge class of Borel sets $\mathbf{\Gamma}$,*

(1) Some set $\mathbb{S}^d_{\mathbf{\Gamma}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus\mathbb{S}^d_{\mathbf{\Gamma}}$ by a pot($\check{\mathbf{\Gamma}}$) set.

(2) If moreover $\mathbf{\Gamma}$ does not have the separation property, and $\mathbf{\Gamma}=\mathbf{\Sigma}^0_\xi$ or $\Delta(\mathbf{\Gamma})$ is a Wadge class, some disjoint sets $\mathbb{S}^0_{\mathbf{\Gamma}},\mathbb{S}^1_{\mathbf{\Gamma}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

Theorem 1.8 *Let T_d be a tree with suitable levels, $\mathbf{\Gamma}$ a non self-dual Wadge class of Borel sets, $(X_i)_{i\in d}$ a sequence of Polish spaces, and A_0, A_1 disjoint analytic subsets of $\Pi_{i\in d}\ X_i$.*

(1) Let $S\in\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus S$ by a pot($\check{\mathbf{\Gamma}}$) set. Then exactly one of the following holds:

(a) The set A_0 is separable from A_1 by a pot($\check{\mathbf{\Gamma}}$) set.

(b) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

(2) Assume moreover that $\mathbf{\Gamma}$ does not have the separation property, and that $\mathbf{\Gamma}=\mathbf{\Sigma}^0_\xi$ or $\Delta(\mathbf{\Gamma})$ is a Wadge class. Let $S^0,S^1\in\mathbf{\Gamma}(\lceil T_d\rceil)$ disjoint not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set. Then exactly one of the following holds:

(a) The set A_0 is separable from A_1 by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

(b) The inequality $\big((d^\omega)_{i\in d},S^0,S^1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

We now come back to the new approach to the study of Wadge classes mentioned earlier. There are a lot of dichotomy results in descriptive set theory about equivalence relations, quasi-orders or even arbitrary Borel or analytic sets. So it is natural to ask for common points to these dichotomies. B. Miller's recent work goes in this direction. He proved many known dichotomies without effective descriptive set theory, using variants of the Kechris-Solecki-Todorčević dichotomy about analytic graphs (see [K-S-T]). Here we want to point out another common point, of effective nature. In these dichotomies, the first possibility of the dichotomy is equivalent to the emptyness of some Σ^1_1 set. For example, in the Kechris-Solecki-Todorčević dichotomy, the Σ^1_1 set is the complement of the union of the Δ^1_1 subsets discrete with respect to the Σ^1_1 graph considered. We prove a strengthening of Theorem 1.8 in which such a Σ^1_1 set appears. We will state in Case (1), unformally. Before that, we need the following notation.

Notation. Let X be a recursively presented Polish space. We denote by Δ_X the topology on X generated by $\Delta^1_1(X)$. This topology is Polish (see (iii) $\Rightarrow$ (i) in the proof of Theorem 3.4 in [Lo3]). The topology τ_1 on $(\omega^\omega)^d$ will be the product topology $\Delta^d_{\omega^\omega}$.

Theorem 1.9 *Let T_d be a tree with Δ^1_1 suitable levels, $\mathbf{\Gamma}$ a non self-dual Wadge class of Borel sets with a Δ^1_1 code, A_0,A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$, and $S\in\mathbf{\Gamma}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus S$ by a pot($\check{\mathbf{\Gamma}}$) set. Then there is a Σ^1_1 subset R of $(\omega^\omega)^d$ such that the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot($\check{\mathbf{\Gamma}}$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot($\check{\mathbf{\Gamma}}$) set.

(c) The set A_0 is not separable from A_1 by a $\check{\mathbf{\Gamma}}(\tau_1)$ set.

(d) $R\neq\emptyset$.

(e) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

This Σ^1_1 set R is build with topologies based on τ_1. The use of the Σ^1_1 set R is the new approach to the study of Wadge classes.

We first prove Theorems 1.7 and 1.8 for the Borel classes, self-dual or not. Then we consider the case of the Wadge classes. In Section 2, we start proving Theorem 1.7. We construct a concrete example of a tree with suitable levels, and we give a general condition to get some complicated sets as in the statement of Theorem 1.7. We actually reduce the problem to a problem in dimension one. In Section 3, we prove Theorem 1.7 for the Borel classes. In Section 4, we prove Theorem 1.8 for the Borel classes, using some tools of effective descriptive set theory and the representation theorem of Borel sets proved in [D-SR]. In Section 5, we prove Theorem 1.7, using the description of Wadge classes in [Lo-SR2]. In Section 6, we prove Theorems 1.3, 1.8 and 1.9. Finally, in Section 7, we give some injectivity complements.

2 A general condition to get some complicated sets

We now build an example of a tree with suitable levels. This tree has to be small enough since we cannot have a reduction on the whole product. But as the same time it has to be big enough to ensure the existence of complicated sets, as in Theorem 1.7.

Notation. Let $b:\omega\rightarrow\omega^2$ be the natural bijection. More precisely, we set, for $l\in\omega$,

$$M(l):=\max\{m\in\omega \mid \Sigma_{k\leq m}\ k\leq l\}.$$

Then we define $b(l)=\big((l)_0,(l)_1\big):=\big(M(l)-l+(\Sigma_{k\leq M(l)}\ k),l-(\Sigma_{k\leq M(l)}\ k)\big)$. One can check that $<n,p>:=b^{-1}(n,p)=(\Sigma_{k\leq n+p}\ k)+p$. More concretely, we get

$$b[\omega]=\{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\}.$$

In the introduction, we mentionned the idenfication between $(d^l)^d$ and $(d^d)^l$, for $l\leq\omega$. More specifically, the bijection we use is given by $\vec{\alpha}\mapsto\Big(\big(\alpha_i(j)\big)_{i\in d}\Big)_{j\in l}$.

Definition 2.1 *We say that $E\subseteq\bigcup_{l\in\omega}\ (d^l)^d$ is an $effective\ frame$ if*

(a) $\forall l\in\omega\ \exists!(s^i_l)_{i\in d}\in E\cap(d^l)^d$.

(b) $\forall p,q,r\in\omega\ \forall t\in d^{<\omega}\ \exists N\in\omega\ (s^i_q it0^N)_{i\in d}\in E$, $(|s^0_q 0t0^N|-1)_0=p$ *and* $\big((|s^0_q 0t0^N|-1)_1\big)_0=r$.

(c) $\forall l>0\ \exists q<l\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s^i_l=s^i_q it$.

(d) The map $l\mapsto(s^i_l)_{i\in d}$ can be coded by a recursive map from ω into ω^d.

We will call T_d the $tree\ on\ d^d\ associated\ with\ an\ effective\ frame$ $E=\{(s^i_l)_{i\in d}\mid l\in\omega\}$:

$$T_d:=\big\{\vec{s}\in(d^d)^{<\omega}\mid(\forall i\in d\ \ s_i=\emptyset)\vee\big(\exists l\in\omega\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s_i=s^i_l it\wedge\forall n<|s_0|\ s_0(n)\leq n\big)\big\}.$$

The uniqueness condition in (a) and Condition (c) ensure that T_d is small enough, and also the almost acyclicity. The definition of T_d ensures that T_d has finite levels. Note that $\mathcal{T}^l=T_d\cap(d^d)^l$ can be coded by a Π^0_1 subset of $(\omega^\omega)^l$ when $d=\omega$. The existence condition in (a) and Condition (b) ensure that T_d is big enough. More specifically, if (X,τ) is a Polish space and σ a finer Polish topology on X, then there is a dense G_δ subset of (X,τ) on which τ and σ coincide. The first part of Condition (b) ensures the possibility to get inside products of dense G_δ sets. The examples in Theorem 1.7 are build using the examples in [Lo-SR1] and [Lo-SR2]. Conditions on verticals are involved, and the second part of Condition (b) gives a control on the choice of verticals. The very last part of Condition (b) is not necessary to get Theorem 1.7 for the Borel classes, but is useful to get Theorem 1.7 for the Wadge classes of Borel sets. Definition 2.1 strengthens Definition 3.1 in [L7], with this very last part of Condition (b), with Condition (d) (ensuring the regularity of the levels of the tree), and also with the last part of the definition of the tree (ensuring the finiteness of the levels of the tree).

Proposition 2.2 *The tree T_d associated with an effective frame is a tree with Δ^1_1 suitable levels. In particular, $\lceil T_d\rceil$ is compact.*

Proof. Let $l \in \omega$. Let us prove that $\mathcal{T}^l$ is Δ^1_1 and finite. We argue by induction on l. The result is clear for $l \leq 1$ since $\mathcal{T}^0 = \{\vec{\emptyset}\}$ and $\mathcal{T}^1 = \{(i)_{i\in d}\}$. If $l \geq 1$ and $\vec{s} \in (d^d)^{<\omega}$, then

$$\vec{s} \in \mathcal{T}^l \ \Leftrightarrow\ |s_0| = l \wedge \exists q < l\ \ \exists t \in d^{<\omega}\ \ \forall i \in d\ \ s_i = s^i_q it \wedge \forall n < l\ \ s_0(n) \leq n.$$

But there are only finitely many possibilities for t since $s_0(n) \leq n$ for each $n < l$, which implies that $t(m) \leq q+1+m < l+1+l$ if $m < |t|$. This implies that $\mathcal{T}^l$ is Δ^1_1 and finite.

• Let $\tilde{T}_d$ be the tree generated by the effective frame:

$$\tilde{T}_d := \big\{\vec{s} \in (d^d)^{<\omega} \mid (\forall i \in d\ \ s_i = \emptyset) \vee \big(\exists l \in \omega\ \exists t \in d^{<\omega}\ \forall i \in d\ \ s_i = s^i_l it\big)\big\}.$$

As $T_d \subseteq \tilde{T}_d$, we get, with obvious notation, $\mathcal{T}^l \subseteq \tilde{\mathcal{T}}^l$ for each integer l. So it is enough to prove that $\tilde{\mathcal{T}}^l$ is one-sided and almost acyclic since these properties are hereditary.

• Let us prove that $\tilde{\mathcal{T}}^l$ is almost acyclic. We argue by induction on l. The result is clear for $l \leq 1$. So assume that $l \geq 1$. We set, for $j \in d$,

$$C_j := \big\{(s^i_q it)_{i\in d} \in \tilde{\mathcal{T}}^{l+1} \mid t \neq \emptyset \wedge t(|t|-1) = j\big\}.$$

We have $\tilde{\mathcal{T}}^{l+1} = \{(s^i_l i)_{i\in d}\} \cup \bigcup_{j\in d} C_j$, and this union is disjoint.

The restriction of $G^{\tilde{\mathcal{T}}^{l+1}}$ to each C_j is isomorphic to $G^{\tilde{\mathcal{T}}^l}$. The other possible $G^{\tilde{\mathcal{T}}^{l+1}}$-edges are between $(s^i_l i)_{i\in d}$ and some vertices in some C_j's. If a $G^{\tilde{\mathcal{T}}^{l+1}}$-cycle exists, we may assume that it involves only $(s^i_l i)_{i\in d}$ and members of some fixed C_{j_0}. But if $\vec{s} \in C_{j_0}$ is $G^{\tilde{\mathcal{T}}^{l+1}}$-related to $(s^i_l i)_{i\in d}$, then we must have $s^{j_0}_l j_0 = s_{j_0}$. This implies the existence of $k < m < n$ showing that $\tilde{\mathcal{T}}^{l+1}$ is almost acyclic.

• Now assume that $\vec{x} \neq \vec{y} \in \tilde{\mathcal{T}}^l$, $i, j \in d$, $x_i = y_i$ and $x_j = y_j$. Then we can write $\vec{x} = (s^i_q it)_{i\in d}$ and $\vec{y} = (s^i_{q'} it')_{i\in d}$ since $\vec{x} \neq \vec{y}$. As $x_i = y_i$, the reverses t^{-1} and $(t')^{-1}$ of t and t' are compatible. If $t = t'$, then $q = |s^i_q| = l-1-|t| = l-1-|t'| = |s^i_{q'}| = q'$ and $\vec{x} = \vec{y}$, which is absurd. Thus $t \neq t'$, for example $|t'| < |t|$, and $t^{-1}(|t'|) = i$. This proves that $i = j$ and $\tilde{\mathcal{T}}^l$ is one-sided.

• Let $\pi_l : \mathcal{T}^{l+1} \to d^d$ defined by $\pi_l(\vec{s}) := \big(s_i(l)\big)_{i\in d}$. As $\mathcal{T}^{l+1}$ is finite, the range c_l of π_l is also finite. Thus $\lceil T_d \rceil$ is compact since $\lceil T_d \rceil \subseteq \Pi_{l\in\omega}\ c_l$. $\square$

We now give an example of an effective frame.

Notation. Let $b_d : \omega \to d^{<\omega}$ be the natural bijection. More specifically,

• If $d < \omega$, then $b_d(0) := \emptyset$ is the sequence of length 0, $b_d(1) := 0$, ..., $b_d(d) := d-1$ are the sequences of length 1 in the lexicographical ordering, and so on.

• If $d = \omega$, then let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $\mathcal{I} : \omega^{<\omega} \to \omega$ defined by $\mathcal{I}(\emptyset) := 1$, and $\mathcal{I}(s) := p_0^{s(0)+1} ... p_{|s|-1}^{s(|s|-1)+1}$ if $s \neq \emptyset$. Note that $\mathcal{I}$ is one-to-one, so that there is an increasing bijection $\imath : \mathrm{Seq} := \mathcal{I}[\omega^{<\omega}] \to \omega$. We set $b_\omega := (\imath \circ \mathcal{I})^{-1} : \omega \to \omega^{<\omega}$.

Note that $|b_d(n)|\leq n$ if $n\in\omega$. Indeed, this is clear if $d<\omega$. If $d=\omega$, then

$$\mathcal{I}\big(b_\omega(n)|0\big)<\mathcal{I}\big(b_\omega(n)|1\big)<\ldots<\mathcal{I}\big(b_\omega(n)\big),$$

so that $(\imath\circ\mathcal{I})\big(b_\omega(n)|0\big)<(\imath\circ\mathcal{I})\big(b_\omega(n)|1\big)<\ldots<(\imath\circ\mathcal{I})\big(b_\omega(n)\big)=n$. This implies that $|b_\omega(n)|\leq n$.

Lemma 2.3 *There is a concrete example of an effective frame.*

Proof. Fix $i\in d$. We set $s^i_0=\emptyset$, and $s^i_{l+1}:=s^i_{(((l)_1)_1)_0}\ i\ b_d\Bigg(\bigg(\Big(((l)_1\Big)_1\bigg)_1\Bigg)\ 0^{l-(((l)_1)_1)_0-|b_d((((l)_1)_1)_1)|}$. Note that $(l)_0+(l)_1=M(l)\leq\Sigma_{k\leq M(l)}\ k\leq l$, so that s^i_l is well defined and $|s^i_l|=l$, by induction on l. It remains to check that Condition (b) in the definition of an effective frame is fullfilled. Set $n:=b_d^{-1}(t)$, $s:=\big\langle\ r,<q,n>\big\rangle$ and $l:=<p,s>$. It remains to put $N:=l-q-|t|$: $(s^i_q it0^N)_{i\in d}=(s^i_{l+1})_{i\in d}$. $\square$

The previous lemma is essentially identical to Lemma 3.3 in [L7]. Now we come to the lemma crucial for the proof of Theorem 1.7. It strengthens Lemma 3.4 in [L7], even if the proof is essentially the same.

Notation. If $s\in\omega^{<\omega}$ and $q\leq|s|$, then $s-s|q$ is defined by $s=(s|q)(s-s|q)$. We extend this definition to the case $s\in\omega^\omega$ when $q<\omega$. In particular, we denote $s^*:=s-s|1$ when $\emptyset\neq s\in\omega^{\leq\omega}$. If $\emptyset\neq s\in\omega^{<\omega}$, then we define $s^-:=s|(|s|-1)$.

• We now define $p:\omega^{<\omega}\setminus\{\emptyset\}\rightarrow\omega$. The definition of $p(s)$ is by induction on $|s|$:

$$p(s):=\begin{cases} s(0)\text{ if }|s|=1,\\ \\ <p(s^-),s(|s|-1)>\ \text{ otherwise.}\end{cases}$$

Note that $p_{|\omega^n}:\omega^n\rightarrow\omega$ is a bijection, for each $n\geq 1$.

• Let $l\leq\omega$ be an ordinal. The map $\Delta:d^l\times d^l\rightarrow 2^l$ is the symmetric difference. So, for $m\in l$,

$$(s\Delta t)(m):=\Delta(s,t)(m)=1\ \Leftrightarrow\ s(m)\neq t(m).$$

• By convention, $\omega-1:=\omega$.

Lemma 2.4 *Let T_d be the tree associated with an effective frame and, for each $i\in d$, G_i a dense G_δ subset of $\Pi''_i\lceil T_d\rceil$. Then there are $\alpha_0\in G_0$ and $F:2^\omega\rightarrow\Pi_{0<i<d}\ G_i$ continuous such that, for $\alpha\in 2^\omega$,*
(a) $\big(\alpha_0,F(\alpha)\big)\in\lceil T_d\rceil$.
(b) For each $s\in\omega^{<\omega}$, and each $m\in\omega$,
(i) $\alpha\big(p(sm)\big)=1\ \Rightarrow\ \exists m'\in\omega\ \ \big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sm')+1\big)=1$.
(ii) $\big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sm)+1\big)=1\ \Rightarrow\ \exists m'\in\omega\ \ \alpha\big(p(sm')\big)=1$.
Moreover, there is an increasing bijection

$$B_\alpha:\{m\in\omega\mid\alpha(m)=1\}\rightarrow\{m'\in\omega\mid\big(\alpha_0\Delta F_0(\alpha)\big)(m'+1)=1\}$$

such that $(m)_0=\big(B_\alpha(m)\big)_0$ *and* $\big((m)_1\big)_0=\Big(\big(B_\alpha(m)\big)_1\Big)_0$ *if* $\alpha(m)=1$.

Proof. Let $(O^i_q)_{q\in\omega}$ be a non-increasing sequence of dense open subsets of $\Pi''_i\lceil T_d\rceil$ whose intersection is G_i. We construct finite approximations of α_0 and F. The idea is to linearize the binary tree $2^{<\omega}$. So we will use the bijection b_2 defined before Lemma 2.3. To construct $F(\alpha)$ we have to imagine, for each length l, the different possibilities for $\alpha|l$. More precisely, we construct a map $l:2^{<\omega}\rightarrow\omega\setminus\{0\}$ satisfying the following conditions:

$$(1)\ \forall t\in 2^{<\omega}\ \ \forall i\in d\ \ (i\leq |t|\ \Rightarrow\ \emptyset\neq N_{s^i_{l(t)}}\cap\Pi''_i\lceil T_d\rceil\subseteq O^i_{|t|}),$$

$$(2)\ \exists v_\emptyset\in d^{<\omega}\ \ \forall i\in d\ \ s^i_{l(\emptyset)}=iv_\emptyset,$$

$$(3)\ \forall t\in 2^{<\omega}\ \ \forall\varepsilon\in 2\ \ \exists v_{t\varepsilon}\in d^{<\omega}\ \ \forall i\in d\ \ s^i_{l(t\varepsilon)}=s^i_{l(t)}(i\cdot\varepsilon)v_{t\varepsilon},$$

$$(4)\ \forall r\in\omega\ \ s^0_{l(b_2(r))}0\subseteq s^0_{l(b_2(r+1))}\wedge\forall t\in 2^{<\omega}\ \ \forall n<l(t)\ \ s^0_{l(t)}(n)\leq n,$$

$$(5)\ \forall t\in 2^{<\omega}\ \ \big(l(t)-1\big)_0=(|t|)_0\wedge\Big(\big(l(t)-1\big)_1\Big)_0=\big((|t|)_1\big)_0.$$

• Assume that this construction is done. As $s^0_{l(0^q)}\subsetneq s^0_{l(0^{q+1})}$ for each integer q, we can define $\alpha_0:=\text{sup}_{q\in\omega}\ s^0_{l(0^q)}$. Similarly, as $s^{i+1}_{l(\alpha|q)}\subsetneq s^{i+1}_{l(\alpha|(q+1))}$, we can define, for $\alpha\in 2^\omega$ and $i<d-1$,

$$F_i(\alpha):=\text{sup}_{q\in\omega}\ s^{i+1}_{l(\alpha|q)},$$

and F is continuous.

(a) Fix $q\in\omega$. We have to see that $\big(\alpha_0,F(\alpha)\big)|q\in T_d$. Note first that $l(t)\geq |t|$ since $l(t\varepsilon)>l(t)$. Then note that $s^0_{l(t)}\subseteq\alpha_0$ since $s^0_{l(0^{|t|})}\subseteq s^0_{l(t)}\subseteq s^0_{l(0^{|t|+1})}$. Thus $\big(\alpha_0,F(\alpha)\big)|l(\alpha|q)=(s^i_{l(\alpha|q)})_{i\in d}\in E$. This implies that $\big(\alpha_0,F(\alpha)\big)|l(\alpha|q)\in T_d$ since $s^0_{l(\alpha|q)}(n)\leq n$ if $n<l(\alpha|q)$. We are done since $l(\alpha|q)\geq q$.

Moreover, $\alpha_0\in\bigcap_{q\in\omega}\ N_{s^0_{l(0^q)}}\cap\Pi''_0\lceil T_d\rceil\subseteq\bigcap_{q\in\omega}\ O^0_q=G_0$. Similarly,

$$F_i(\alpha)\in\bigcap_{q\in\omega}\ N_{s^{i+1}_{l(\alpha|q)}}\cap\Pi''_{i+1}\lceil T_d\rceil\subseteq\bigcap_{q\geq i+1}\ O^{i+1}_q=G_{i+1}.$$

(b).(i) We set $t:=\alpha|p(sm)$, so that $s^1_{l(t)}\ 1\subseteq s^1_{l(t1)}=s^1_{l(\alpha|(p(sm)+1))}\subseteq F_0(\alpha)$. As $\big(l(t)-1\big)_0=p(s)$ (or $(m)_0$ if $s=\emptyset$), there is m' with $l(t)=p(sm')+1$ (or $l(t)=m'+1$ and $(m')_0=(m)_0$ if $s=\emptyset$). But $s^0_{l(t)}\ 0\subseteq s^0_{l(\alpha|(p(sm)+1))}\subseteq\alpha_0$, so that $\alpha_0\big(l(t)\big)\neq F_0(\alpha)\big(l(t)\big)$.

(ii) First notice that the only coordinates where α_0 and $F_0(\alpha)$ can differ are 0 and the $l(\alpha|q)$'s. Therefore there is an integer q with $p(sm)+1=l(\alpha|q)$. In particular, $(q)_0=\big(l(\alpha|q)-1\big)_0=p(s)$ (or $(m)_0$ if $s=\emptyset$). Thus there is m' with $q=p(sm')$ (or $q=m'$ and $(m')_0=(m)_0$ if $s=\emptyset$). We have $\alpha_0\big(l(\alpha|q)\big)=s^0_{l(\alpha|(q+1))}\big(l(\alpha|q)\big)=0\neq F_0(\alpha)\big(l(\alpha|q)\big)=s^1_{l(\alpha|(q+1))}\big(l(\alpha|q)\big)=\alpha(q)$. So $\alpha(q)=1$ and $\alpha\big(p(sm')\big)=1$.

Now it is clear that the formula $B_\alpha(m):=l(\alpha|m)-1$ defines the bijection we are looking for.

• So let us prove that the construction is possible. We construct $l(t)$ by induction on $b_2^{-1}(t)$.

As $(i0^\infty)_{i\in d}\in\lceil T_d\rceil$, $0^\infty\in\Pi_0''\lceil T_d\rceil$ and O_0^0 is not empty. Thus there is $u_0^0\in d^{<\omega}\setminus\{\emptyset\}$ such that $\emptyset\neq N_{u_0^0}\cap\Pi_0''\lceil T_d\rceil\subseteq O_0^0$. Choose $\beta_0\in N_{u_0^0}\cap\Pi_0''\lceil T_d\rceil$, and $\vec{\alpha}\in\lceil T_d\rceil$ such that $\alpha_0=\beta_0$. Then $\vec{\alpha}||u_0^0|\in T_d$ and $u_0^0(n)\leq n$ for each $n<|u_0^0|$. Note that $u_0^0(0)=0$ and $(u_0^0-u_0^0|1)(n)=u_0^0(n+1)\leq 1+n$ for each $n<|u_0^0|-1$. We choose $N_\emptyset\in\omega$ with $\big(i\ (u_0^0-u_0^0|1)\ 0^{N_\emptyset}\big)_{i\in d}\in E$, $(|0\ (u_0^0-u_0^0|1)\ 0^{N_\emptyset}|-1)_0=(0)_0$ and $\big((|0\ (u_0^0-u_0^0|1)\ 0^{N_\emptyset}|-1)_1\big)_0=\big((0)_1\big)_0$. We put $v_\emptyset:=(u_0^0-u_0^0|1)\ 0^{N_\emptyset}$ and $l(\emptyset):=|0\ (u_0^0-u_0^0|1)\ 0^{N_\emptyset}|$.

As $(iv_\emptyset 0^\infty)_{i\in d}\in\lceil T_d\rceil$, $N_{0v_\emptyset 0}\cap\Pi_0''\lceil T_d\rceil$ is a nonempty open subset of $\Pi_0''\lceil T_d\rceil$. Thus there is $u_1^0\in d^{<\omega}$ such that $\emptyset\neq N_{0v_\emptyset 0u_1^0}\cap\Pi_0''\lceil T_d\rceil\subseteq O_1^0$. As before we see that $u_1^0(n)\leq 1+|v_\emptyset|+1+n$ for each $n<|u_1^0|$. This implies that $(iv_\emptyset 0u_1^0 0^\infty)_{i\in d}\in\lceil T_d\rceil$. Thus $N_{1v_\emptyset 0u_1^0}\cap\Pi_1''\lceil T_d\rceil$ is a nonempty open subset of $\Pi_1''\lceil T_d\rceil$. So there is $u_1^1\in d^{<\omega}$ such that $\emptyset\neq N_{1v_\emptyset 0u_1^0u_1^1}\cap\Pi_1''\lceil T_d\rceil\subseteq O_1^1$. Choose $\beta_1\in N_{1v_\emptyset 0u_1^0u_1^1}\cap\Pi_1''\lceil T_d\rceil$, and $\vec{\gamma}\in\lceil T_d\rceil$ such that $\gamma_1=\beta_1$. Then $\vec{\gamma}||1v_\emptyset 0u_1^0u_1^1|\in T_d$ and $\gamma_0(n)\leq n$ for each $n<|1v_\emptyset 0u_1^0u_1^1|$. This implies that $\gamma_0(|1v_\emptyset 0u_1^0|+n)\leq|1v_\emptyset 0u_1^0|+n$ for each $n<|u_1^1|$. But $u_1^1(n)$ is either 1, or $\gamma_0(|1v_\emptyset 0u_1^0|+n)$. Thus $u_1^1(n)\leq|1v_\emptyset 0u_1^0|+n$ if $n<|u_1^1|$. We choose $N_0\in\omega$ such that $(s^i_{l(\emptyset)}\ 0u_1^0u_1^1\ 0^{N_0})_{i\in d}\in E$, $\big(l(\emptyset)+|u_1^0u_1^1|+N_0\big)_0=(1)_0$ and $\Big(\big(l(\emptyset)+|u_1^0u_1^1|+N_0\big)_1\Big)_0=\big((1)_1\big)_0$. We put $v_0:=u_1^0u_1^1\ 0^{N_0}$ and $l(0):=l(\emptyset)+1+|v_0|$.

Assume that $\big(l(t)\big)_{b_2^{-1}(t)\leq r}$ satisfying (1)-(5) have been constructed, which is the case for $r=1$. Fix $t\in 2^{<\omega}$ and $\varepsilon\in 2$ such that $b_2(r+1)=t\varepsilon$, with $r\geq 1$. Note that $b_2^{-1}(t)<r$, so that $l(t)<l\big(b_2(r)\big)$, by induction assumption.

As $N_{s^0_{l(b_2(r))}}\cap\Pi_0''\lceil T_d\rceil$ is nonempty, $N_{s^0_{l(b_2(r))}0}\cap\Pi_0''\lceil T_d\rceil$ is nonempty too. Thus there is $u^0_{|t|+1}$ in $d^{<\omega}$ such that $\emptyset\neq N_{s^0_{l(b_2(r))}0u^0_{|t|+1}}\cap\Pi_0''\lceil T_d\rceil\subseteq O^0_{|t|+1}$. As before we see that $u^0_{|t|+1}(n)\leq l\big(b_2(r)\big)+1+n$ for each $n<|u^0_{|t|+1}|$. Arguing as in the case $r=1$, we prove, for each $1\leq i\leq|t|+1$, the existence of $u^i_{|t|+1}\in d^{<\omega}$ such that $\emptyset\neq N_{s^i_{l(t)}(i\cdot\varepsilon)(s^0_{l(b_2(r))}-s^0_{l(b_2(r))}|(l(t)+1))0u^0_{|t|+1}\ldots u^i_{|t|+1}}\cap\Pi_i''\lceil T_d\rceil\subseteq O^i_{|t|+1}$ and $u^i_{|t|+1}(n)\leq l\big(b_2(r)\big)+1+|u^0_{|t|+1}\ldots u^{i-1}_{|t|+1}|+n$ for each $n<|u^i_{|t|+1}|$ ($u^i_{|t|+1}(n)$ can be i, in which case we use the fact that $l(t)\geq|t|$). We choose $N_{t\varepsilon}\in\omega$ such that

$$\left(s^i_{l(t)}\ (i\cdot\varepsilon)\ \left(s^0_{l(b_2(r))}-s^0_{l(b_2(r))}|\big(l(t)+1\big)\right)\ 0\ u^0_{|t|+1}\ldots u^{|t|+1}_{|t|+1}\ 0^{N_{t\varepsilon}}\right)_{i\in d}\in E,$$

$\left(l\big(b_2(r)\big)+|u^0_{|t|+1}\ldots u^{|t|+1}_{|t|+1}|+N_{t\varepsilon}\right)_0=(|t|+1)_0$ and

$$\left(\left(l\big(b_2(r)\big)+|u^0_{|t|+1}\ldots u^{|t|+1}_{|t|+1}|+N_{t\varepsilon}\right)_1\right)_0=\big((|t|+1)_1\big)_0.$$

We put $l(t\varepsilon):=l(t)+1+|v_{t\varepsilon}|$, where by definition

$$v_{t\varepsilon}:=\left(s^0_{l(b_2(r))}-s^0_{l(b_2(r))}|\big(l(t)+1\big)\right)\ 0\ u^0_{|t|+1}\ldots u^{|t|+1}_{|t|+1}\ 0^{N_{t\varepsilon}}.$$

This finishes the proof. $\square$

Now we come to the general condition to get some complicated sets as in the statement of Theorem 1.7 announced in the introduction.

Notation. The map $\mathcal{S}\!:\!2^\omega\!\rightarrow\!2^\omega$ is the shift map: $\mathcal{S}(\alpha)(m)\!:=\!\alpha(m\!+\!1)$.

Definition 2.5 *We say that $C\!\subseteq\!2^\omega$ is $compatible\ with\ comeager\ sets$ (ccs for short) if*

$$\alpha\!\in\!C\ \Leftrightarrow\ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\!\in\!C,$$

for each $\alpha_0\!\in\!d^\omega$ and $F\!:\!2^\omega\!\rightarrow\!(d^\omega)^{d-1}$ satisfying the conclusion of Lemma 2.4.(b).

Notation. Let T_d be the tree associated with an effective frame, and $C\!\subseteq\!2^\omega$. We put

$$S^d_C\!:=\!\big\{\vec{\alpha}\!\in\!\lceil T_d\rceil\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\!\in\!C\big\}.$$

Lemma 2.6 *Let T_d be the tree associated with an effective frame, and $\mathbf{\Gamma}$ a non self-dual Wadge class of Borel sets.*
(1) Assume that C is a $\mathbf{\Gamma}$-complete ccs set. Then $S^d_C\!\in\!\mathbf{\Gamma}(\lceil T_d\rceil)$ is a Borel subset of $(d^\omega)^d$, and is not separable from $\lceil T_d\rceil\!\setminus\!S^d_C$ by a pot$(\check{\mathbf{\Gamma}})$ set.
(2) Assume that C^0, $C^1\!\in\!\mathbf{\Gamma}$ are disjoint, ccs, and not separable by a $\Delta(\mathbf{\Gamma})$ set. Then $S^d_{C^0},S^d_{C^1}$ are in $\mathbf{\Gamma}(\lceil T_d\rceil)$, disjoint Borel subsets of $(d^\omega)^d$, and not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

Proof. (1) It is clear that $S^d_C\in\mathbf{\Gamma}(\lceil T_d\rceil)$ since $\mathcal{S}$ and Δ are continuous. So S^d_C is a Borel subset of $(d^\omega)^d$ since $\lceil T_d\rceil$ is a closed subset of $(d^\omega)^d$. Indeed, $\lceil T_\omega\rceil$ is closed:

$$\vec{\alpha}\!\in\!\lceil T_\omega\rceil\ \ \Leftrightarrow\ \ \forall n\!\in\!\omega\!\setminus\!\{0\}\ \ \exists l\!<\!n\ \ \forall i\!\in\!\omega\ \ s^i_l i\!\subseteq\!\alpha_i\wedge(\alpha_i|n\!-\!s^i_l i)\!=\!(\alpha_0|n\!-\!s^0_l 0)\wedge\alpha_0(n)\!\leq\!n.$$

We argue by contradiction to see that S^d_C is not separable from $\lceil T_d\rceil\!\setminus\!S^d_C$ by a pot$(\check{\mathbf{\Gamma}})$ set: this gives $P\in$ pot$(\check{\mathbf{\Gamma}})$. For each $i\in d$ there is a dense G_δ subset G_i of the compact space $\Pi''_i\lceil T_d\rceil$ such that $P\cap(\Pi_{i\in d}\ G_i)\!\in\!\check{\mathbf{\Gamma}}(\Pi_{i\in d}\ G_i)$, and $S^d_C\cap(\Pi_{i\in d}\ G_i)\!\subseteq\!P\cap(\Pi_{i\in d}\ G_i)\!\subseteq\!(\Pi_{i\in d}\ G_i)\!\setminus\!(\lceil T_d\rceil\!\setminus\!S^d_C)$.

Lemma 2.4 provides $\alpha_0\!\in\!G_0$ and $F\!:\!2^\omega\!\rightarrow\!\Pi_{0<i<d}\ G_i$ continuous. Let

$$D\!:=\!\big\{\alpha\!\in\!2^\omega\mid\big(\alpha_0,F(\alpha)\big)\!\in\!P\cap(\Pi_{i\in d}\ G_i)\big\}.$$

Then $D\!\in\!\check{\mathbf{\Gamma}}$. Let us prove that $C\!=\!D$, which will contradict the fact that $C\!\notin\!\check{\mathbf{\Gamma}}$. As C is ccs, $\alpha\!\in\!C$ is equivalent to $\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\!\in\!C$. Thus

$$\alpha\!\in\!C\Rightarrow\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\!\in\!C\Rightarrow\big(\alpha_0,F(\alpha)\big)\!\in\!S^d_C\cap(\Pi_{i\in d}\ G_i)\!\subseteq\!P\cap(\Pi_{i\in d}\ G_i)\Rightarrow\alpha\!\in\!D.$$

Similarly, $\alpha\!\notin\!C\Rightarrow\alpha\!\notin\!D$, and $C\!=\!D$.

(2) We argue as in (1). $\square$

This lemma reduces the problem of finding some complicated sets as in the statement of Theorem 1.7 to a problem in dimension 1.

3 The proof of Theorem 1.7 for the Borel classes

The full version of Theorem 1.7 for the Borel classes is as follows:

Theorem 3.1 *We can find concrete examples of a tree T_d with Δ^1_1 suitable levels, together with, for each $1\leq\xi<\omega_1$,*
(1) Some set $\mathbb{S}^d_\xi\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus\mathbb{S}^d_\xi$ by a pot($\mathbf{\Pi}^0_\xi$) set.
(2) Some disjoint sets $\mathbb{S}^0_\xi,\mathbb{S}^1_\xi\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ not separable by a pot($\mathbf{\Delta}^0_\xi$) set.

This is an application of Lemma 2.6. We now introduce the objects useful to define the suitable sets C's of this lemma. These objects will also be useful in the general case. The following definition can be found in [Lo-SR2] (see Definition 2.2).

Definition 3.2 *A set $H\subseteq 2^\omega$ is $\mathbf{\Gamma}$-$strategically\ complete$ if*
(a) $H\in\mathbf{\Gamma}(2^\omega)$.
(b) If $A\in\mathbf{\Gamma}(\omega^\omega)$, then Player 2 wins the Wadge game $G(A,H)$ (where Player 1 plays $\alpha\in\omega^\omega$, Player 2 plays $\beta\in 2^\omega$ and Player 2 wins if $\alpha\in A\Leftrightarrow\beta\in H$).

The following definition can essentially be found in [Lo-SR1] (see Section 3) and [Lo-SR2] (see Definition 2.3).

Definition 3.3 *Let $\eta<\omega_1$. A function $\rho:2^\omega\rightarrow 2^\omega$ is an $independent\ \eta$-$function$ if*
(a) For some function $\pi:\omega\rightarrow\omega$, the value $\rho(\alpha)(m)$, for each $\alpha\in 2^\omega$ and each integer m, depends only on the values of α on $\pi^{-1}(\{m\})$.
(b) For each integer m, we set $\mathcal{C}_m:=\{\alpha\in 2^\omega\mid\rho(\alpha)(m)=1\}$.
(1) If $\eta=0$, then for each integer m the set $\mathcal{C}_m$ is a $\mathbf{\Delta}^0_1$-complete set.
(2) If $\eta=\theta+1$ is successor, then for each integer m the set $\mathcal{C}_m$ is a $\mathbf{\Pi}^0_{1+\theta}$-strategically complete set.
(3) If η limit, then for some sequence $(\theta_m)_{m\in\omega}$ with $\theta_m<\eta$ and $\sup_{p\geq 1}\theta_{m_p}=\eta$ for each one-to-one sequence $(m_p)_{p\geq 1}$ of integers, and for each integer m the set $\mathcal{C}_m$ is a $\mathbf{\Pi}^0_{1+\theta_m}$-strategically complete set.

Note that we added a condition when $\eta=0$. Moreover, we do not ask the sequence $(\theta_m)_{m\in\omega}$ to be increasing, unlike in [Lo-SR2], Definition 2.3. Note also that an independent η-function has to be $\mathbf{\Sigma}^0_{1+\eta}$-measurable. Moreover, if ρ is an independent η-function, then π has to be onto.

Examples. In [Lo-SR1], Lemma 3.3, the map $\rho_0:2^\omega\rightarrow 2^\omega$ defined as follows is introduced:

$$\rho_0(\alpha)(m):=\begin{cases}1\text{ if }\alpha(<m,n>)=0\text{, for each }n\in\omega,\\ \\ 0\text{ otherwise.}\end{cases}$$

Then ρ_0 is clearly an independent 1-function, with $\pi(k)=(k)_0$. In this paper, $\rho^\eta_0:2^\omega\rightarrow 2^\omega$ is also defined for $\eta<\omega_1$ as follows, by induction on η (see the proof of Theorem 3.2).

We put

- $\rho_0^0 := \mathrm{Id}_{2^\omega}$.
- $\rho_0^{\theta+1} := \rho_0 \circ \rho_0^\theta$.
- If $\eta > 0$ is limit, then fix a sequence $(\theta_m^\eta)_{m\in\omega} \subseteq \eta$ of successor ordinals with $\Sigma_{m\in\omega}\ \theta_m^\eta = \eta$. We define $\rho_0^{(m,m+1)} : 2^\omega \to 2^\omega$ by

$$\rho_0^{(m,m+1)}(\alpha)(i) := \begin{cases} \alpha(i) \text{ if } i < m, \\ \\ \rho_0^{\theta_m^\eta}\big(\mathcal{S}^m(\alpha)\big)(i-m) \text{ if } i \geq m. \end{cases}$$

We set $\rho_0^{(0,m+1)} := \rho_0^{(m,m+1)} \circ \rho_0^{(m-1,m)} \circ \ldots \circ \rho_0^{(0,1)}$ and $\rho_0^\eta(\alpha)(m) := \rho_0^{(0,m+1)}(\alpha)(m)$. The authors prove that ρ_0^η is an independent η-function (see the proof of Theorem 3.2). In this paper, the set $H_{1+\eta} := (\rho_0^\eta)^{-1}(\{0^\infty\})$ is also introduced, and the authors prove that $H_{1+\eta}$ is $\mathbf{\Pi}^0_{1+\eta}$-complete (see Theorem 3.2).

Notation. Let $1 \leq \xi := 1+\eta < \omega_1$. We set $C_\xi := \neg H_\xi$. If moreover $\varepsilon \in 2$, then we set

$$C_\xi^\varepsilon := \big\{\alpha \in 2^\omega \mid \exists m \in \omega\ \ \rho_0^\eta(\alpha)(m) = 1 \wedge \forall l < m\ \ \rho_0^\eta(\alpha)(l) = 0 \wedge (m)_0 \equiv \varepsilon \ (\text{mod } 2)\big\}.$$

Then we set $\mathbb{S}_\xi^d := S^d_{C_\xi}$ and $\mathbb{S}_\xi^\varepsilon := S^d_{C_\xi^\varepsilon}$.

Theorem 3.1 is a corollary of Proposition 2.2, Lemmas 2.3 and 2.6, and of the following lemma.

Lemma 3.4 *Let $1 \leq \xi < \omega_1$.*
(1) The set C_ξ is a $\mathbf{\Sigma}^0_\xi$-complete ccs set.
(2) The sets $C_\xi^0, C_\xi^1 \in \mathbf{\Sigma}^0_\xi$, are disjoint, ccs, and not separable by a $\mathbf{\Delta}^0_\xi$ set.

Proof. (1) C_ξ is $\mathbf{\Sigma}^0_\xi$-complete since H_ξ is $\mathbf{\Pi}^0_\xi$-complete.

• Assume that α_0, F satisfy the conclusion of Lemma 2.4.(b). Let us prove that

$$\rho_0^\eta(\alpha) = \rho_0^\eta\Big(\mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big)\Big),$$

for each $1 \leq \eta < \omega_1$ and $\alpha \in 2^\omega$. For $\eta = 1$ we apply the conclusion of Lemma 2.4.(b) to $s \in \omega$. Then we have, by induction, $\rho_0^{\theta+1}(\alpha) = \rho_0\big(\rho_0^\theta(\alpha)\big) = \rho_0\left(\rho_0^\theta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)\right) = \rho_0^{\theta+1}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$. From this we deduce, for $\lambda > 0$ limit, by induction again, that

$$\rho_0^{(0,1)}(\alpha) = \rho_0^{\theta_0^\lambda}(\alpha) = \rho_0^{\theta_0^\lambda}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big) = \rho_0^{(0,1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big).$$

Thus $\rho_0^{(0,m+1)}(\alpha) = \rho_0^{(0,m+1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$, and

$$\rho_0^\lambda(\alpha)(m) = \rho_0^{(0,m+1)}(\alpha)(m) = \rho_0^{(0,m+1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m) = \rho_0^\lambda\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m).$$

• If we apply the previous point, or the conclusion of Lemma 2.4.(b) to $s := \emptyset$, then we get

$$\alpha \in C_\xi \Leftrightarrow \exists m \in \omega\ \ \rho_0^\eta(\alpha)(m) = 1 \Leftrightarrow \exists m' \in \omega\ \ \rho_0^\eta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m') = 1 \Leftrightarrow \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big) \in C_\xi.$$

Thus C_ξ is ccs.

(2) Note first that $C^0_\xi, C^1_\xi\in\mathbf{\Sigma}^0_\xi$ since ρ^η_0 is $\mathbf{\Sigma}^0_{1+\eta}$-measurable, are clearly disjoint, and are ccs as in (1) since $(m)_0=\big(B_\alpha(m)\big)_0$ in Lemma 2.4.(b).

• We set, for $\varepsilon\in 2$, $V_\varepsilon:=\big\{\alpha\in 2^\omega\mid\exists m\in\omega\ \ \rho^\eta_0(\alpha)(m)=1\ \text{ and }\ (m)_0\equiv\varepsilon\ (\text{mod }2)\big\}$. Then V_ε is a $\mathbf{\Sigma}^0_\xi$ set since ρ^η_0 is $\mathbf{\Sigma}^0_{1+\eta}$-measurable. Let us prove that V_ε is $\mathbf{\Sigma}^0_\xi$-complete.

- If $\eta=0$, then $0^\infty\in\overline{V_\varepsilon}\setminus V_\varepsilon$, so that V_ε is $\mathbf{\Sigma}^0_1$-complete.

- If $\eta=\theta+1$, then ρ^η_0 is an independent η-function. Let $(A_m)_{m\in\omega}$ be a sequence of $\mathbf{\Pi}^0_{1+\theta}(2^\omega)$ sets. Choose a continuous map $f_m:2^\omega\to 2^\omega$ such that $A_m=f_m^{-1}(\mathcal{C}_m)$. We define $f:2^\omega\to 2^\omega$ by $f(\alpha)(k):=f_m(\alpha)(k)$ if $\pi_\eta(k)=m$, and f is continuous. Moreover,

$$\alpha\in A_m\Leftrightarrow f_m(\alpha)\in\mathcal{C}_n\Leftrightarrow f(\alpha)\in\mathcal{C}_m,$$

so that $\bigcup_{m\in\omega,(m)_0\equiv\varepsilon\ (\text{mod }2)}\ A_m=f^{-1}(V_\varepsilon)$. Thus V_ε is $\mathbf{\Sigma}^0_\xi$-complete.

- If η is the limit of the θ_m's, then ρ^η_0 is an independent η-function. We argue as in the successor case to see that V_ε is $\mathbf{\Sigma}^0_\xi$-complete.

• We argue by contradiction, which gives $D\in\mathbf{\Delta}^0_\xi$ separating C^0_ξ from C^1_ξ. Let v_0, v_1 be disjoint $\mathbf{\Sigma}^0_\xi$ subsets of 2^ω. Then we can find a continuous map $f_\varepsilon:2^\omega\to 2^\omega$ such that $v_\varepsilon=f_\varepsilon^{-1}(V_\varepsilon)$. As ρ^η_0 is an independent η-function, we get $\pi_\eta:\omega\to\omega$. We define a map $f:2^\omega\to 2^\omega$ by $f(\alpha)(k):=f_\varepsilon(\alpha)(k)$ if $\big(\pi_\eta(k)\big)_0\equiv\varepsilon$ (mod 2), and f is continuous. Note that $\alpha\in v_\varepsilon\Leftrightarrow f_\varepsilon(\alpha)\in V_\varepsilon\Leftrightarrow f(\alpha)\in V_\varepsilon$, so that $v_\varepsilon=f^{-1}(V_\varepsilon)$. Thus $\alpha\in v_0\Leftrightarrow f(\alpha)\in V_0\Leftrightarrow f(\alpha)\in V_0\setminus V_1\subseteq C^0_\xi\subseteq D$ since v_0 is disjoint from v_1. Similarly, $\alpha\in v_1\Leftrightarrow f(\alpha)\in V_1\setminus V_0\subseteq C^1_\xi\subseteq\neg D$. Thus $f^{-1}(D)$ separates v_0 from v_1. As $f^{-1}(D)\in\mathbf{\Delta}^0_\xi$, this implies that $\mathbf{\Sigma}^0_\xi$ has the separation property, which contradicts 22.C in [K]. $\square$

4 The proof of Theorem 1.8 for the Borel classes

The full versions of Theorem 1.8 and Corollary 1.9 for the Borel classes are as follows:

Theorem 4.1 *Let T_d be a tree with suitable levels, $1\leq\xi<\omega_1$, $(X_i)_{i\in d}$ a sequence of Polish spaces, and A_0, A_1 disjoint analytic subsets of $\Pi_{i\in d}\ X_i$.*

(1) Let $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Pi}^0_\xi$) set.

(b) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

If we moreover assume that S is not separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_\xi$) set, then this is a dichotomy.

(2) Let $S^0,S^1\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Delta}^0_\xi$) set.

(b) The inequality $\big((d^\omega)_{i\in d},S^0,S^1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

If we moreover assume that S^0 is not separable from S^1 by a pot($\mathbf{\Delta}^0_\xi$) set, then this is a dichotomy.

Corollary 4.2 *Let $\mathbf{\Gamma}$ be Borel class. Then there are Borel subsets $\mathbb{S}^0$, $\mathbb{S}^1$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Gamma}$) set.

(b) The inequality $\big((d^\omega)_{i\in d},\mathbb{S}^0,\mathbb{S}^1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

4.1 Acyclicity

In this subsection we prove a result that will be used later to show Theorem 4.1. This is the place where the essence of the notion of a finite one-sided almost acyclic set is really used.

Lemma 4.1.1 *Assume that $\mathcal{T}\subseteq\mathcal{X}^d$ is finite. Then the following are equivalent:*

(a) The set $\mathcal{T}$ is one-sided and almost acyclic.

(b) For each $\overrightarrow{x^0}\in\mathcal{T}$, there is an integer $0\neq\mathcal{L}<d+2$ and a partition $(M_j)_{j\in\mathcal{L}}$ of $\mathcal{T}\setminus\{\overrightarrow{x^0}\}$ with

(1) $\forall i\in d\ \ \forall j\neq k\in\mathcal{L}\ \ \Pi_i[M_j]\cap\Pi_i[M_k]=\emptyset$.

(2) $\forall i\in d\ \ \forall j\in\mathcal{L}\ \ \forall\vec{x}\in M_j\ \ x_i=x^0_i\ \Rightarrow\ i=j$.

Proof. (a) $\Rightarrow$ (b) If $\vec{y}\neq\vec{z}\in\mathcal{T}$ and $\left(\overrightarrow{y^j}\right)_{j\leq l}$ is a walk in $G^{\mathcal{T}}$ with $\overrightarrow{y^0}=\vec{y}$ and $\overrightarrow{y^l}=\vec{z}$, then we choose such a walk of minimal length, and we call it $w_{\vec{y},\vec{z}}$. We will define a partition of $\mathcal{T}$. We put, for $j\in d$,

$$N := \{\ \vec{x}\in\mathcal{T}\mid\vec{x}\neq\overrightarrow{x^0}\wedge w_{\vec{x},\overrightarrow{x^0}}\text{ does not exist }\},$$
$$L_j := \{\ \vec{x}\in\mathcal{T}\mid\vec{x}\neq\overrightarrow{x^0}\wedge\big(w_{\vec{x},\overrightarrow{x^0}}(|w_{\vec{x},\overrightarrow{x^0}}|-2)\big)_j=x^0_j\ \}.$$

So we defined a partition $\big(N,(L_j)_{j\in d}\big)$ of $\mathcal{T}\setminus\{\overrightarrow{x^0}\}$ since $\mathcal{T}$ is one-sided. As $\mathcal{T}$ is finite, there is $j_0\in d$ minimal such that $L_j=\emptyset$ if $j>j_0$. We set $M_j:=L_j$ if $j\leq j_0$, $M_{j_0+1}:=N$ and $\mathcal{L}:=j_0+2$.

(1) Let us prove that $\Pi_i[L_j]\cap\Pi_i[N]=\emptyset$, for each $i,j\in d$. We argue by contradiction. This gives $x_i\in\Pi_i[L_j]\cap\Pi_i[N]$, $\vec{x}\in L_j$, and also $\vec{y}\in N$ such that $x_i=y_i$. As $\vec{x},\vec{y}\in\mathcal{T}$ and $L_j\cap N=\emptyset$, $\vec{x}\neq\vec{y}$ and $\vec{x},\vec{y}$ are $G^{\mathcal{T}}$-related. Note that $w_{\vec{y},\overrightarrow{x^0}}$ does not exist, and that $w_{\vec{x},\overrightarrow{x^0}}$ exists. Now the sequence $\left(\vec{y},\vec{x},...,\overrightarrow{x^0}\right)$ shows the existence of $w_{\vec{y},\overrightarrow{x^0}}$, which is absurd.

It remains to see that $\Pi_i[L_j]\cap\Pi_i[L_k]=\emptyset$, for each $i,j,k\in d$ with $j\neq k$. We argue by contradiction. This gives $x_i\in\Pi_i[L_j]\cap\Pi_i[L_k]$, $\vec{x}\in L_j$, and also $\vec{y}\in L_k$ such that $x_i=y_i$. As $\vec{x},\vec{y}\in\mathcal{T}$ and $j\neq k$, $\vec{x}\neq\vec{y}$ and $\vec{x},\vec{y}$ are $G^{\mathcal{T}}$-related. Let us denote $w_{\vec{x},\overrightarrow{x^0}}:=\left(\overrightarrow{z^n}\right)_{n\leq l+1}$ and $w_{\vec{y},\overrightarrow{x^0}}:=\left(\overrightarrow{y^n}\right)_{n\leq l'+1}$. Note that $\overrightarrow{z^l}\neq\overrightarrow{y^{l'}}$ since $z^l_j=x^0_j$ and $y^{l'}_j\neq x^0_j$, since otherwise $\overrightarrow{y^{l'}}$, $\overrightarrow{x^0}\in\mathcal{T}$, $\overrightarrow{y^{l'}}\neq\overrightarrow{x^0}$ and $y^{l'}_j=x^0_j$, $y^{l'}_k=x^0_k$, which contradicts the fact that $\mathcal{T}$ is one-sided.

We denote by $W:=\left(\overrightarrow{w^n}\right)_{n\leq L}$ the following $G^{\mathcal{T}}$-walk: $\overrightarrow{z^l},\overrightarrow{z^{l-1}},...,\overrightarrow{z^0},\overrightarrow{y^0},\overrightarrow{y^1},...,\overrightarrow{y^{l'}}$. If there are $k<n\leq L$ with $\overrightarrow{w^k}=\overrightarrow{w^n}$, then we put $W':=\overrightarrow{w^0},...,\overrightarrow{w^k},\overrightarrow{w^{n+1}},...,\overrightarrow{w^L}$. If we iterate this construction, then we get a $G^{\mathcal{T}}$-walk without repetition $V:=\left(\overrightarrow{v^n}\right)_{n\leq L'}$ from $\overrightarrow{w^0}$ to $\overrightarrow{w^L}$.

If there are $i \in d$ and $k+1<n\leq L'$ with $v_i^k=v_i^n$, then we put $V':=\overrightarrow{v^0},...,\overrightarrow{v^k},\overrightarrow{v^n},...,\overrightarrow{v^{L'}}$. If we iterate this construction, then we get a $G^{\mathcal{T}}$-walk without repetition $U:=\left(\overrightarrow{u^n}\right)_{n\leq L''}$ from $\overrightarrow{w^0}$ to $\overrightarrow{w^L}$ for which it is not possible to find $i\in d$ and $k+1<n\leq L''$ with $u_i^k=u_i^n$.

Now $\overrightarrow{x^0},\overrightarrow{u^0},...,\overrightarrow{u^{L''}},\overrightarrow{x^0}$ is a $G^{\mathcal{T}}$-cycle contradicting the almost acyclicity of $\mathcal{T}$.

(2) If $\vec{x}\in N$, then $w_{\vec{x},\overrightarrow{x^0}}$ does not exist. This implies that $x_i\neq x_i^0$ for each $i\in d$, since otherwise $\vec{x}$ and $\overrightarrow{x^0}$ would be $G^{\mathcal{T}}$-related, which contradicts the non-existence of $w_{\vec{x},\overrightarrow{x^0}}$.

If $\vec{x}\in L_j$, then i is the only coordinate for which $x_i=x_i^0$ since $\mathcal{T}$ is one-sided. Note that $w_{\vec{x},\overrightarrow{x^0}}=\left(\vec{x},\overrightarrow{x^0}\right)$. As $\vec{x}\in L_j$, we get $\left(w_{\vec{x},\overrightarrow{x^0}}(|w_{\vec{x},\overrightarrow{x^0}}|-2)\right)_j=x_j^0$. But $w_{\vec{x},\overrightarrow{x^0}}(|w_{\vec{x},\overrightarrow{x^0}}|-2)=\vec{x}$. Thus $x_j=x_j^0$ and $i=j$.

(b) $\Rightarrow$ (a) Let $\overrightarrow{x^0}\neq\vec{x}\in\mathcal{T}$, $i,j\in d$ such that $x_i^0=x_i$ and $x_j^0=x_j$, and $k\in\mathcal{L}$ such that $\vec{x}\in M_k$. By (2) we get $i=k=j$ and $\mathcal{T}$ is one-sided. Now consider a $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n\leq L}$. By (1) there is $j\in\mathcal{L}$ such that $\overrightarrow{x^n}\in M_j$ for each $0<n<L$. Then by (2) we get $x_j^0=x_j^1=x_j^{L-1}$ and $\mathcal{T}$ is almost acyclic. $\square$

Definition 4.1.2 and Lemma 4.1.3 below are essentially due to G. Debs (see Subsection 2.1 in [L7]):

Definition 4.1.2 *(Debs) Let* $\Theta:\mathcal{X}^d\rightarrow 2^{(\omega^\omega)^d}$, $\mathcal{T}\subseteq\mathcal{X}^d$. *We say that the map* $\theta=\Pi_{i\in d}\ \theta_i\in\left((\omega^\omega)^{\mathcal{X}}\right)^d$ *is a* $\pi\text{-}selector\ on\ \mathcal{T}\ for\ \Theta$ *if*
(a) $\theta(\vec{x})=\left(\theta_i(x_i)\right)_{i\in d}$ *for each* $\vec{x}\in\mathcal{X}^d$.
(b) $\theta(\vec{x})\in\Theta(\vec{x})$ *for each* $\vec{x}\in\mathcal{T}$.

Lemma 4.1.3 *(Debs) Let l be an integer,* $\mathcal{X}:=d^{l+1}$, $\mathcal{T}\subseteq\mathcal{X}^d$ *be* Δ^1_1, *finite, one-sided, and almost acyclic,* $\Theta:\mathcal{X}^d\rightarrow\Sigma^1_1\left((\omega^\omega)^d\right)$, *and* $\overline{\Theta}:\mathcal{X}^d\rightarrow\Sigma^1_1\left((\omega^\omega)^d\right)$ *defined by* $\overline{\Theta}(\vec{x}):=\overline{\Theta(\vec{x})}^{\tau_1}$. *Then* Θ *admits a* π*-selector on* $\mathcal{T}$ *if* $\overline{\Theta}$ *does.*

Proof. (a) Let $\overrightarrow{x^0}\in\mathcal{T}$, and $\Psi:\mathcal{X}^d\rightarrow\Sigma^1_1\left((\omega^\omega)^d\right)$. We assume that $\Psi(\vec{x})=\Theta(\vec{x})$ if $\vec{x}\neq\overrightarrow{x^0}$, and that $\Psi\left(\overrightarrow{x^0}\right)\subseteq\overline{\Theta\left(\overrightarrow{x^0}\right)}^{\tau_1}$. We first prove that Θ admits a π-selector on $\mathcal{T}$ if Ψ does.

• Lemma 4.1.1 gives a finite partition $(M_j)_{j\in\mathcal{L}}$ of $\mathcal{T}\setminus\{\overrightarrow{x^0}\}$. Fix a π-selector $\tilde{\psi}$ on $\mathcal{T}$ for Ψ, and let $M:=\max\ (d\cap\mathcal{L})$. We define Σ^1_1 sets U_i, for $i\leq M$, by

$$U_i:=\left\{\alpha\in\omega^\omega\mid\exists\psi\in\left((\omega^\omega)^{\mathcal{X}}\right)^d\ \ \alpha=\psi_i(x_i^0)\wedge\forall\vec{x}\in\mathcal{T}\ \ \psi(\vec{x})\in\Psi(\vec{x})\right\}.$$

As $\tilde{\psi}\left(\overrightarrow{x^0}\right)=\left(\tilde{\psi}_i(x_i^0)\right)_{i\in d}\in\Psi\left(\overrightarrow{x^0}\right)\cap\left((\Pi_{i\leq M}\ U_i)\times(\omega^\omega)^{d-M-1}\right)$ we get

$$\emptyset\neq\Psi\left(\overrightarrow{x^0}\right)\cap\left((\Pi_{i\leq M}\ U_i)\times(\omega^\omega)^{d-M-1}\right)\subseteq\overline{\Theta\left(\overrightarrow{x^0}\right)}^{\tau_1}\cap\left((\Pi_{i\leq M}\ U_i)\times(\omega^\omega)^{d-M-1}\right).$$

By the separation theorem this implies that $\Theta\left(\overrightarrow{x^0}\right)\cap\left((\Pi_{i\leq M}\ U_i)\times(\omega^\omega)^{d-M-1}\right)$ is not empty and contains some point $\vec{\alpha}$. Fix $i\leq M$. As $\alpha_i\in U_i$ there is $\psi^i\in\left((\omega^\omega)^{\mathcal{X}}\right)^d$ such that $\alpha_i=\psi^i_i(x^0_i)$ and $\psi^i(\vec{x})\in\Psi(\vec{x})$ if $\vec{x}\in\mathcal{T}$.

• Now we can define $\theta_i:\mathcal{X}\rightarrow\omega^\omega$, for each $i\in d$. We put

$$\theta_i(x_i):=\begin{cases}\alpha_i \text{ if } x_i=x^0_i,\\ \\ \psi^j_i(x_i) \text{ if } x_i\in\Pi_i[M_j]\setminus\{x^0_i\}\wedge j\leq M,\\ \\ \psi^0_i(x_i) \text{ otherwise.}\end{cases}$$

Then we set $\theta(\vec{x})(i):=\theta_i(x_i)$ if $i\in d$.

• It remains to see that $\theta(\vec{x})\in\Theta(\vec{x})$ for each $\vec{x}\in\mathcal{T}$.

Note that $\theta\left(\overrightarrow{x^0}\right)=\vec{\alpha}\in\Theta\left(\overrightarrow{x^0}\right)$. So we may assume that $\vec{x}\neq\overrightarrow{x^0}$. So let $j\in\mathcal{L}$ with $\vec{x}\in M_j$.

- If $x_i\neq x^0_i$ for each $i\in d$ and $j\leq M$, then $\theta(\vec{x})=\left(\theta_i(x_i)\right)_{i\in d}=\psi^j(\vec{x})\in\Psi(\vec{x})=\Theta(\vec{x})$.

- Similarly, if $x_i\neq x^0_i$ for each $i\in d$ and $j>M$, then $\theta(\vec{x})=\left(\theta_i(x_i)\right)_{i\in d}=\psi^0(\vec{x})\in\Psi(\vec{x})=\Theta(\vec{x})$.

- If $x_i=x^0_i$ for some $i\in d$, then $i=j\leq M$. This implies that $\theta_j(x_j)=\alpha_j=\psi^j_j(x^0_j)=\psi^j_j(x_j)$ and

$$\theta(\vec{x})=\left(\theta_i(x_i)\right)_{i\in d}=\psi^j(\vec{x})\in\Psi(\vec{x})=\Theta(\vec{x}).$$

(b) Write $\mathcal{T}:=\left\{\overrightarrow{x^1},\ldots,\overrightarrow{x^n}\right\}$, and set $\Psi_0:=\overline{\Theta}$. We define $\Psi_{j+1}:\mathcal{X}^d\rightarrow\varSigma^1_1\left((\omega^\omega)^d\right)$ as follows. We put $\Psi_{j+1}(\vec{x}):=\Psi_j(\vec{x})$ if $\vec{x}\neq\overrightarrow{x^{j+1}}$, and $\Psi_{j+1}\left(\overrightarrow{x^{j+1}}\right):=\Theta\left(\overrightarrow{x^{j+1}}\right)$, for $j<n$. The result now follows from an iterative application of (a). $\square$

4.2 The topologies

In this subsection we prove two other results that will be used to show Theorem 4.1. We use tools of effective descriptive set theory (the reader should see [M] for the basic notions). We first recall a classical result in the spirit of Theorem 3.3.1 in [H-K-Lo].

Notation. Let X be a recursively presented Polish space. Using the bijection between ω and ω^2 defined before Definition 2.1, we can build a bijection $(x_n)\mapsto<x_n>$ between $(X^\omega)^\omega$ and X^ω by the formula $<x_n>(l):=x_{(l)_0}\left((l)_1\right)$. The inverse map $x\mapsto\left((x)_n\right)$ is given by $(x)_n(p):=x(<n,p>)$. These bijections are recursive.

Lemma 4.2.1 *Let X be a recursively presented Polish space. Then there are $\varPi^1_1$ sets $W^X\subseteq\omega^\omega$, $C^X\subseteq\omega^\omega\times X$ with $\{(\alpha,x)\in\omega^\omega\times X\mid\alpha\in W^X$ and $x\notin C^X_\alpha\}\in\varPi^1_1$, $\varDelta^1_1(X)=\{C^X_\alpha\mid\alpha\in\varDelta^1_1\cap W^X\}$, and $\boldsymbol{\Delta}^1_1(X)=\{C^X_\alpha\mid\alpha\in W^X\}$.*

Proof. By 3E.2, 3F.6 and 3H.1 in [M], there is $\mathcal{U}^X\in\Pi^1_1(\omega^\omega\times X)$ which is universal for $\mathbf{\Pi}^1_1(X)$ and satisfies the two following properties:

- A subset P of X is Π^1_1 if and only if there is $\alpha\in\omega^\omega$ recursive with $P=\mathcal{U}^X_\alpha$.

- There is $S^X:\omega^\omega\times\omega^\omega\rightarrow\omega^\omega$ recursive such that $(\alpha,\beta,x)\in\mathcal{U}^{\omega^\omega\times X}\Leftrightarrow\big(S^X(\alpha,\beta),x\big)\in\mathcal{U}^X$.

We set, for $\varepsilon\in 2$, $U_\varepsilon:=\{(\alpha,x)\in\omega^\omega\times X\mid\big((\alpha)_\varepsilon,x\big)\in\mathcal{U}^X\}$. Then $U_\varepsilon\in\Pi^1_1$. By 4B.10 in [M], Π^1_1 has the reduction property, which gives $U'_0,U'_1\in\Pi^1_1$ disjoint with $U'_\varepsilon\subseteq U_\varepsilon$ and $U'_0\cup U'_1=U_0\cup U_1$. We set $W^X:=\{\alpha\in\omega^\omega\mid(U'_0)_\alpha\cup(U'_1)_\alpha=X\}$ and $C^X:=U'_0$, which defines Π^1_1 sets. Moreover,

$$\alpha\in W^X\wedge x\notin C^X_\alpha\Leftrightarrow\alpha\in W^X\wedge(\alpha,x)\in U'_1$$

is Π^1_1 in (α,x). Assume that $A\in\Delta^1_1(X)$, which gives $\alpha_0,\alpha_1\in\omega^\omega$ recursive with $A=\mathcal{U}^X_{\alpha_0}$ (resp., $\neg A=\mathcal{U}^X_{\alpha_1}$). We define $\alpha\in\omega^\omega$ by $(\alpha)_\varepsilon:=\alpha_\varepsilon$, so that α is recursive. We get

$$\begin{array}{l}x\in A\Leftrightarrow(\alpha_0,x)\in\mathcal{U}^X\Leftrightarrow(\alpha,x)\in U_0\Leftrightarrow(\alpha,x)\in U_0\setminus U_1\Leftrightarrow(\alpha,x)\in U'_0,\\ x\notin A\Leftrightarrow(\alpha_1,x)\in\mathcal{U}^X\Leftrightarrow(\alpha,x)\in U_1\Leftrightarrow(\alpha,x)\in U_1\setminus U_0\Leftrightarrow(\alpha,x)\in U'_1,\end{array}$$

so that $\alpha\in W^X$ and $C^X_\alpha=A$. This also proves that $\mathbf{\Delta}^1_1(X)\subseteq\{C^X_\alpha\mid\alpha\in W^X\}$.

Conversely, let $\alpha\in\Delta^1_1\cap W^X$. Then $C^X_\alpha\in\Pi^1_1$, and $x\notin C^X_\alpha\Leftrightarrow\alpha\in W^X$ and $x\notin C^X_\alpha$, so that $\neg C^X_\alpha\in\Pi^1_1$ and $C^X_\alpha\in\Delta^1_1$. Note that this also proves that $\mathbf{\Delta}^1_1(X)\supseteq\{C^X_\alpha\mid\alpha\in W^X\}$. $\square$

We now give some notation to state an effective version of Theorem 4.1.

Notation. Let X be a recursively presented Polish space.

• We will use the Gandy-Harrington topology Σ_X on X generated by $\Sigma^1_1(X)$. Recall that the set $\Omega_X:=\{x\in X\mid\omega^x_1=\omega^{\text{CK}}_1\}$ is Borel and Σ^1_1, that (Ω_X,Σ_X) is a 0-dimensional Polish space (the intersection of Ω_X with any nonempty Σ^1_1 set is a nonempty clopen subset of (Ω_X,Σ_X)) (see [L8]).

• Recall the topology τ_1 defined before Theorem 1.9. We will also consider some topologies between τ_1 and $\Sigma_{(\omega^\omega)^d}$. Let $2\leq\xi<\omega^{\text{CK}}_1$. The topology τ_ξ is generated by $\Sigma^1_1\big((\omega^\omega)^d\big)\cap\mathbf{\Pi}^0_{<\xi}(\tau_1)$. We have $\mathbf{\Sigma}^0_1(\tau_\xi)\subseteq\mathbf{\Sigma}^0_\xi(\tau_1)$, so that $\mathbf{\Pi}^0_1(\tau_\xi)\subseteq\mathbf{\Pi}^0_\xi(\tau_1)$. These topologies are similar to the ones considered in [Lo2] (see Definition 1.5).

• We set pot($\mathbf{\Pi}^0_0$) $:=\{\Pi_{i\in d}\ A_i\mid A_i\in\mathbf{\Delta}^1_1(\omega^\omega)$, and $A_i=\omega^\omega$ for almost every $i\in d\}$. We also set $W:=W^{(\omega^\omega)^d}$ and $C:=C^{(\omega^\omega)^d}$ (see Lemma 4.2.1). We will define specifically, for $\xi<\omega_1$,

$$\big\{(\beta,\gamma)\in\omega^\omega\times W\mid\beta\text{ codes a pot}(\mathbf{\Pi}^0_\xi)\text{ set and }C_\gamma\text{ is the set coded by }\beta\big\}.$$

The way we will do it is not the simplest possible (we can in fact forget β, and work with γ integer instead of real, see [L7]). We do it this way to start to give the flavor of what is going on with the Wadge classes.

• To do this, we set

$$V_0 := \left\{ (\beta,\gamma)\in\omega^\omega\times W \mid \forall i<\beta(0)\ \ (\beta^*)_i\in W^{\omega^\omega}\wedge\gamma\in\Delta^1_1(\beta)\wedge \left[\begin{array}{l} \beta(0)=d\wedge C_\gamma=\Pi_{i<\beta(0)}\ C^{\omega^\omega}_{(\beta^*)_i}\ \text{if}\ d<\omega \\ \\ C_\gamma=\Big(\Pi_{i<\beta(0)}\ C^{\omega^\omega}_{(\beta^*)_i}\Big)\times(\omega^\omega)^\omega\ \text{if}\ d=\omega \end{array}\right.\right\}.$$

We define an inductive operator Φ over $\omega^\omega\times\omega^\omega$ (see [C]) as follows:

$$\Phi(A):=A\cup V_0\ \cup\big\{(\beta,\gamma)\in\omega^\omega\times W\mid\gamma\in\Delta^1_1(\beta)\wedge \exists\gamma'\in\Delta^1_1(\beta)\ \ \forall n\in\omega\ \ \big((\beta)_n,(\gamma')_n\big)\in A\wedge\neg C_\gamma=\bigcup_{n\in\omega}\ C_{(\gamma')_n}\big\}.$$

Then Φ is clearly a Π^1_1 monotone inductive operator. We set, for any ordinal ξ, $V_\xi:=\Phi^\xi$ (which is coherent with the definition of V_0). We also set $V_{<\xi}:=\bigcup_{\eta<\xi}\ V_\eta$. The effective version of Theorem 4.1, which is the specific version of Theorem 1.9 for the Borel classes, is as follows:

Theorem 4.2.2 *Let T_d be a tree with Δ^1_1 suitable levels, $1\leq\xi<\omega_1^{CK}$, and A_0, A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$.*

(1) Assume that $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ is not separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_\xi$) set. Then the following are equivalent:

(a) The set A_0 is not separable from A_1 by a pot($\mathbf{\Pi}^0_\xi$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot($\mathbf{\Pi}^0_\xi$) set.

(c) $\neg\big(\exists(\beta,\gamma)\in(\Delta^1_1\times\Delta^1_1)\cap V_\xi\ \ A_0\subseteq C_\gamma\subseteq\neg A_1\big)$.

(d) The set A_0 is not separable from A_1 by a $\mathbf{\Pi}^0_\xi(\tau_1)$ set.

(e) $\overline{A_0}^{\tau_\xi}\cap A_1\neq\emptyset$.

(f) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

(2) The sets V_ξ and $V_{<\xi}$ are Π^1_1.

(3) Assume that $S^0,S^1\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ are disjoint and not separable by a pot($\mathbf{\Delta}^0_\xi$) set. Then the following are equivalent:

(a) The set A_0 is not separable from A_1 by a pot($\mathbf{\Delta}^0_\xi$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot($\mathbf{\Delta}^0_\xi$) set.

(c) $\neg\big(\exists(\beta,\gamma),(\beta',\gamma')\in(\Delta^1_1\times\Delta^1_1)\cap V_\xi\ \ C_{\gamma'}=\neg C_\gamma$ and $A_0\subseteq C_\gamma\subseteq\neg A_1\big)$.

(d) The set A_0 is not separable from A_1 by a $\mathbf{\Delta}^0_\xi(\tau_1)$ set.

(e) $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}\neq\emptyset$.

(f) The inequality $\big((d^\omega)_{i\in d},S^0,S^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

The proofs of Theorems 4.1 and 4.2.2 will be by induction on ξ. This appears in the statement of the following lemma.

Lemma 4.2.3 *(1) The set V_0 is $\varPi^1_1$.*

(2) Let $1\leq\xi<\omega_1^{CK}$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$.
(a) The set $V_{<\xi}$ is $\varPi^1_1$.
(b) Fix $A\in\varSigma^1_1\big((\omega^\omega)^d\big)$. Then $\overline{A}^{\tau_\xi}\in\varSigma^1_1\big((\omega^\omega)^d\big)$.
(c) Let $n\geq 1$, $1\leq\xi_1<\xi_2<\ldots<\xi_n\leq\xi$, and S_1, $\ldots$, S_n be $\varSigma^1_1$ sets. Assume that $S_i\subseteq\overline{S_{i+1}}^{\tau_{\xi_{i+1}}}$ for $1\leq i<n$. Then $S_n\cap\bigcap_{1\leq i<n}\ \overline{S_i}^{\tau_{\xi_i}}$ is τ_1-dense in $\overline{S_1}^{\tau_1}$.

Proof. (1) The set V_0 is clearly $\varPi^1_1$.

(2).(a) The proof is contained in the proof of Theorem 4.1 in [L7]. It is a consequence of Lemma 4.8 in [C].

(b) The proof is essentially the proof of Lemma 2.2.2.(a) in [L7].

(c) The proof is essentially the proof of Lemma 2.2.2.(b) in [L7]. $\square$

Lemma 4.2.4 *Let $S,T\in\varSigma^1_1\big((\omega^\omega)^d\big)$ such that S is τ_1-dense in T, $(X_i)_{i\in d}$ a sequence of $\varSigma^1_1$ subsets of ω^ω such that $X_i=\omega^\omega$ if $i\geq i_0$. Then $S\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $T\cap(\Pi_{i\in d}\ X_i)$.*

Proof. Let $(\Delta_i)_{i\in d}$ be a sequence of $\varDelta^1_1$ subsets of ω^ω such that $\Delta_i=\omega^\omega$ if $i\geq j_0\geq i_0$, and also $T\cap(\Pi_{i\in d}\ I_i)\neq\emptyset$, where $I_i:=X_i\cap\Delta_i$. We have to see that $S\cap(\Pi_{i\in d}\ I_i)\neq\emptyset$. We argue by contradiction. This gives a sequence $(D_i)_{i\in d}$ of $\varDelta^1_1$ subsets of ω^ω such that $I_i\subseteq D_i$ if $i\in d$, and $S\cap(\Pi_{i\in d}\ D_i)=\emptyset$, by j_0 applications of the separation theorem. But $T\cap(\Pi_{i\in d}\ D_i)\neq\emptyset$, and $D_i=\omega^\omega$ if $i\geq j_0$. So $S\cap(\Pi_{i\in d}\ D_i)\neq\emptyset$, by τ_1-density of S in T, which is absurd. $\square$

4.3 Representation of Borel sets

Now we come to the representation theorem of Borel sets by G. Debs and J. Saint Raymond (see [D-SR]). It specifies the classical result of Lusin asserting that any Borel set in a Polish space is the bijective continuous image of a closed subset of the Baire space. The material in this Subsection can be found in Subsection 2.3 of [L7], but we recall most of it since it will be used iteratively in the case of Wadge classes. The following definition can be found in [D-SR].

Definition 4.3.1 *(Debs-Saint Raymond) Let c be a countable set. A partial order relation R on $c^{<\omega}$ is a $tree\ relation$ if, for $t\in c^{<\omega}$,*
(a) $\emptyset\ R\ t$.
(b) The set $P_R(t):=\{s\in c^{<\omega}\mid s\ R\ t\}$ is finite and linearly ordered by R.

For instance, the non strict extension relation $\subseteq$ is a tree relation.

• *Let R be a tree relation. An R-$branch$ is an $\subseteq$-maximal subset of $c^{<\omega}$ linearly ordered by R. We denote by $[R]$ the set of all infinite R-branches.*

We equip $(c^{<\omega})^\omega$ with the product of the discrete topology on $c^{<\omega}$. If R is a tree relation, then the space $[R]\subseteq(c^{<\omega})^\omega$ is equipped with the topology induced by that of $(c^{<\omega})^\omega$. The map $h:c^\omega\rightarrow[\subseteq]$ defined by $h(\gamma):=(\gamma|j)_{j\in\omega}$ is an homeomorphism.

• *Let R, S be tree relations with $R\subseteq S$. The* $canonical\ map$ $\Pi\colon[R]\to[S]$ *is defined by*

$$\Pi(\mathcal{B}):=\textit{the unique } S\textit{-branch containing } \mathcal{B}.$$

• *Let S be a tree relation. We say that $R\subseteq S$ is* $distinguished$ *in S if*

$$\forall s,t,u\in c^{<\omega}\qquad \left.\begin{array}{c} s\ S\ t\ S\ u\\ s\ R\ u\end{array}\right\}\Rightarrow s\ R\ t.$$

For example, let C be a closed subset of c^ω, and define

$$s\ R\ t\ \Leftrightarrow\ s\subseteq t\wedge N_s\cap C\neq\emptyset.$$

Then R is distinguished in $\subseteq$.

• *Let $\eta<\omega_1$. A family $(R^{(\rho)})_{\rho\leq\eta}$ of tree relations is a* $resolution\ family$ *if*

(a) $R^{(\rho+1)}$ is a distinguished subtree of $R^{(\rho)}$, for all $\rho<\eta$.

(b) $R^{(\lambda)}=\bigcap_{\rho<\lambda}\ R^{(\rho)}$, for all limit $\lambda\leq\eta$.

We will use the following extension of the property of distinction:

Lemma 4.3.2 *Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family, and $\rho<\eta$. Assume that $s\ R^{(0)}\ s'\ R^{(\rho)}\ s''$ and $s\ R^{(\rho+1)}\ s''$. Then $s\ R^{(\rho+1)}\ s'$.*

Notation. Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family such that $R^{(0)}$ is a subrelation of $\subseteq$, $\rho\leq\eta$ and $t\in c^{<\omega}\setminus\{\emptyset\}$. We set

$$t^\rho:=t\mid \max\{r<|t|\mid t|r\ R^{(\rho)}\ t\}.$$

We enumerate $\{t^\rho\mid\rho\leq\eta\}$ by $\{t^{\xi_i}\mid 1\leq i\leq n\}$, where $1\leq n\in\omega$ and $\xi_1<\ldots<\xi_n=\eta$. We can write $t^{\xi_n}\subsetneq t^{\xi_{n-1}}\subsetneq\ldots\subsetneq t^{\xi_2}\subsetneq t^{\xi_1}\subsetneq t$. By Lemma 4.3.2 we have $t^{\xi_{i+1}}\ R^{(\xi_i+1)}\ t^{\xi_i}$ for each $1\leq i<n$.

Lemma 4.3.3 *Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family such that $R^{(0)}$ is a subrelation of $\subseteq$, t in $c^{<\omega}\setminus\{\emptyset\}$ and $1\leq i<n$.*

(a) Set $\eta_i:=\{\rho\leq\eta\mid t^{\xi_i}\subseteq t^\rho\}$. Then η_i is a successor ordinal.

(b) We may assume that $t^{\xi_i+1}\subsetneq t^{\xi_i}$.

The following is part of Theorem I-6.6 in [D-SR].

Theorem 4.3.4 *(Debs-Saint Raymond) Let $\eta<\omega_1$, R a tree relation, $(I_n)_{n\in\omega}$ a sequence of $\mathbf{\Pi}^0_{\eta+1}$ subsets of $[R]$. Then there is a resolution family $(R^{(\rho)})_{\rho\leq\eta}$ with*

(a) $R^{(0)}=R$.

(b) The canonical map $\Pi\colon[R^{(\eta)}]\to[R]$ is a continuous bijection.

(c) The set $\Pi^{-1}(I_n)$ is a closed subset of $[R^{(\eta)}]$ for each integer n.

Now we come to the actual proof of Theorem 4.1.

4.4 Proof of Theorem 4.1

The next result is essentially Theorem 2.4.1 in [L7]. But we give its proof since it is the basis for further generalizations.

Theorem 4.4.1 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ a successor ordinal, $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$, and A_0, A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*
(a) $\overline{A_0}^{\tau_\xi}\cap A_1=\emptyset$.
(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Proof. Fix $\eta<\omega_1^{\text{CK}}$ with $\xi=\eta+1$.

• Recall the finite sets c_l defined at the end of the proof of Proposition 2.2 (we only used the fact that T_d has finite levels to see that they are finite). With the notation of Definition 4.3.1, we put $c:=\bigcup_{l\in\omega}\ c_l$, so that c is countable. The set $I:=h[\lceil T_d\rceil\setminus S]$ is a $\mathbf{\Pi}^0_{\eta+1}$ subset of $[\subseteq]$. Theorem 4.3.4 provides a resolution family. We put

$$D:=\big\{\vec{s}\in T_d\mid\vec{s}=\vec{\emptyset}\vee\exists\mathcal{B}\in\Pi^{-1}(I)\ \ \vec{s}\in\mathcal{B}\big\}.$$

• Assume that $\overline{A_0}^{\tau_\xi}\cap A_1$ is not empty. Recall that (Ω_X,Σ_X) is a Polish space (see the notation at the beginning of Section 4.2). We fix a complete metric d_X on (Ω_X,Σ_X).

• We construct
- $(\alpha^i_s)_{i\in d,s\in\Pi''_iT_d}\subseteq\omega^\omega$,
- $(O^i_s)_{i\leq|s|,i\in d,s\in\Pi''_iT_d}\subseteq\Sigma^1_1(\omega^\omega)$,
- $(U_{\vec{s}})_{\vec{s}\in T_d}\subseteq\Sigma^1_1\big((\omega^\omega)^d\big)$.

We want these objects to satisfy the following conditions.

$$(1)\ \alpha^i_s\in O^i_s\subseteq\Omega_{\omega^\omega}\wedge(\alpha^i_{s_i})_{i\in d}\in U_{\vec{s}}\subseteq\Omega_{(\omega^\omega)^d},$$

$$(2)\ O^i_{sq}\subseteq O^i_s,$$

$$(3)\ \text{diam}_{d_{\omega^\omega}}(O^i_s)\leq 2^{-|s|}\wedge\text{diam}_{d_{(\omega^\omega)^d}}(U_{\vec{s}})\leq 2^{-|\vec{s}|},$$

$$(4)\ U_{\vec{s}}\subseteq\overline{A_0}^{\tau_\xi}\cap A_1\text{ if }\vec{s}\in D,$$

$$(5)\ U_{\vec{s}}\subseteq A_0\text{ if }\vec{s}\notin D,$$

$$(6)\ \big(1\leq\rho\leq\eta\wedge\vec{s}\ R^{(\rho)}\ \vec{t}\big)\Rightarrow U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^{\tau_\rho},$$

$$(7)\ \big((\vec{s},\vec{t}\in D\vee\vec{s},\vec{t}\notin D)\wedge\vec{s}\ R^{(\eta)}\ \vec{t}\big)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}.$$

• Let us prove that this construction is sufficient to get the theorem.

- Fix $\vec{\beta}\in\lceil T_d\rceil$. Then we can define $(j_k)_{k\in\omega}:=(j_k^{\vec{\beta}})_{k\in\omega}$ by $\Pi^{-1}\big((\vec{\beta}|j)_{j\in\omega}\big)=(\vec{\beta}|j_k)_{k\in\omega}$, with the inequalities $j_k<j_{k+1}$. In particular, $\vec{\beta}|j_k\ R^{(\eta)}\ \vec{\beta}|j_{k+1}$. We have

$$\vec{\beta}\notin S\ \Leftrightarrow\ h(\vec{\beta})=(\vec{\beta}|j)_{j\in\omega}\in I\ \Leftrightarrow\ (\vec{\beta}|j_k)_{k\in\omega}\in\Pi^{-1}(I)\ \Leftrightarrow\ \forall k\geq k_0:=0\ \ \vec{\beta}|j_k\in D$$

since $\Pi^{-1}(I)$ is a closed subset of $[R^{(\eta)}]$. Similarly, $\vec{\beta}\in S$ is equivalent to the existence of $k_0\in\omega$ such that $\vec{\beta}|j_k\notin D$ for each $k\geq k_0$.

This implies that $(U_{\vec{\beta}|j_k})_{k\geq k_0}$ is a non-increasing sequence of nonempty clopen subsets of the space $(\Omega_{(\omega^\omega)^d},\Sigma_{(\omega^\omega)^d})$ whose $d_{(\omega^\omega)^d}$-diameters tend to zero, and we can define

$$\{\mathcal{F}(\vec{\beta})\}:=\bigcap_{k\geq k_0}\ U_{\vec{\beta}|j_k}\subseteq\Omega_{(\omega^\omega)^d}.$$

Note that $\mathcal{F}(\vec{\beta})$ is the limit of $\big((\alpha^i_{\beta_i|j_k})_{i\in d}\big)_{k\in\omega}$.

- Now let $\gamma\in\Pi''_i\lceil T_d\rceil$, and $\vec{\beta}\in\lceil T_d\rceil$ such that $\beta_i=\gamma$. We set $f_i(\gamma):=\mathcal{F}_i(\vec{\beta})$. This defines $f_i:\Pi''_i\lceil T_d\rceil\rightarrow\omega^\omega$.

Note that $f_i(\gamma)$ is the limit of $(\alpha^i_{\gamma|j})_{j\in\omega}$. Indeed, $f_i(\gamma)$ is the limit of $(\alpha^i_{\gamma|j_k})_{k\in\omega}$. If $j\geq i$, then $\alpha^i_{\gamma|j}\in O^i_{\gamma|j}$, and the sequence $(O^i_{\gamma|j})_{j\geq i}$ is decreasing. Fix $\varepsilon>0$, $k\geq i$ such that $2^{-k}<\varepsilon$. Then we get, if $j\geq k$, $d_{\omega^\omega}\big(f_i(\gamma),\alpha^i_{\gamma|j}\big)\leq\mathrm{diam}_{d_{\omega^\omega}}(O^i_{\gamma|j})\leq 2^{-j}\leq 2^{-k}<\varepsilon$. In particular, $f_i(\gamma)$ does not depend on the choice of $\vec{\beta}$. This also proves that f_i is continuous on $\Pi''_i\lceil T_d\rceil$.

- Note that $\mathcal{F}_i(\vec{\beta})$ is the limit of some subsequence of $(\alpha^i_{\beta_i|j})_{j\in\omega}$, by continuity of the projections. Thus $\mathcal{F}_i(\vec{\beta})=f_i(\beta_i)$, and $\mathcal{F}(\vec{\beta})=(\Pi_{i\in d}\ f_i)(\vec{\beta})$. This implies that the inclusions $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$ and $\lceil T_d\rceil\setminus S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$ hold.

• So let us prove that the construction is possible.

- Let $(\alpha^i_\emptyset)_{i\in d}\in\overline{A_0}^{\tau_\xi}\cap A_1\cap\Omega_{(\omega^\omega)^d}$, which is nonempty since $\overline{A_0}^{\tau_\xi}\cap A_1\neq\emptyset$ is Σ^1_1, by Lemma 4.2.3.(2).(b). Then we choose a Σ^1_1 subset $U_{\vec{\emptyset}}$ of $(\omega^\omega)^d$, with $d_{(\omega^\omega)^d}$-diameter at most 1, such that

$$(\alpha^i_\emptyset)_{i\in d}\in U_{\vec{\emptyset}}\subseteq\overline{A_0}^{\tau_\xi}\cap A_1\cap\Omega_{(\omega^\omega)^d}.$$

We choose a Σ^1_1 subset $O^0_\emptyset$ of ω^ω, with d_{ω^ω}-diameter at most 1, with $\alpha^0_\emptyset\in O^0_\emptyset\subseteq\Omega_{\omega^\omega}$, which is possible since $\Omega_{(\omega^\omega)^d}\subseteq\Omega^d_{\omega^\omega}$. Assume that $(\alpha^i_s)_{|s|\leq l}$, $(O^i_s)_{|s|\leq l}$ and $(U_{\vec{s}})_{|\vec{s}|\leq l}$ satisfying conditions (1)-(7) have been constructed, which is the case for $l=0$.

- Let $\overrightarrow{tm}\in T_d\cap(d^{l+1})^d$. Note that $\overrightarrow{tm}^\eta\in D$ if $\overrightarrow{tm}^\eta\in D$ is not equivalent to $\overrightarrow{tm}\in D$ (see the notation before Lemma 4.3.3).

- The conclusions in the assertions (a), (b) and (c) of the following claim do not really depend on their respective assumptions, but we will use these assertions later in this form. We define $X_i:=O^i_{t_i}$ if $i\leq l$, and ω^ω if $i>l$.

Claim. Assume that $\eta>0$.

(a) The set $A_0\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\eta\in D$ and $\overrightarrow{tm}\notin D$.

(b) The set $U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\eta,\overrightarrow{tm}\in D$ or $\overrightarrow{tm}^\eta,\overrightarrow{tm}\notin D$.

Indeed, let us forget $\Pi_{i\in d}\ X_i$ for the moment. We may assume that $\overrightarrow{tm}^{\xi_i+1}\subsetneq\overrightarrow{tm}^{\xi_i}$ if $1\leq i<n$, by Lemma 4.3.3. We set $S_i:=U_{\overrightarrow{tm}^{\xi_i}}$, for $1\leq\xi_i\leq\eta$. As $\overrightarrow{tm}^{\xi_{i+1}}\ R^{(\xi_i+1)}\ \overrightarrow{tm}^{\xi_i}$, we can write $S_i\subseteq\overline{S_{i+1}}^{\tau_{\xi_i+1}}$, for $1\leq\xi_i<\eta$, by induction assumption. If $\overrightarrow{tm}^\eta\in D$ and $\overrightarrow{tm}\notin D$, then $S_n\subseteq\overline{A_0}^{\tau_{\eta+1}}$. Thus $A_0\cap\bigcap_{1\leq\xi_i\leq\eta}\ \overline{U_{\overrightarrow{tm}^{\xi_i}}}^{\tau_{\xi_i}}$ and $U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\ \overline{U_{\overrightarrow{tm}^{\xi_i}}}^{\tau_{\xi_i}}$ are τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}$, by Lemma 4.2.3.(2).(c).

But if $1\leq\rho\leq\eta$, then there is $1\leq i\leq n$ with $\overrightarrow{tm}^\rho=\overrightarrow{tm}^{\xi_i}$. And $\rho\leq\xi_i$ since we have $\overrightarrow{tm}^{\xi_i+1}\subsetneq\overrightarrow{tm}^{\xi_i}$ if $1\leq i<n$. We are done since $\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}=\bigcap_{1\leq\xi_i\leq\eta}\ \overline{U_{\overrightarrow{tm}^{\xi_i}}}^{\tau_{\xi_i}}$ and

$$U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}=U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\ \overline{U_{\overrightarrow{tm}^{\xi_i}}}^{\tau_{\xi_i}}$$

The claim now comes from Lemma 4.2.4. $\diamond$

- Let $\mathcal{X}:=d^{l+1}$. The map $\Theta:\mathcal{X}^d\to\varSigma^1_1\big((\omega^\omega)^d\big)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Theta(\overrightarrow{tm}):=\begin{cases}A_0\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{(\omega^\omega)^d}\text{ if }\overrightarrow{tm}^\eta\in D\wedge\overrightarrow{tm}\notin D,\\ \\ U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\text{ if }\overrightarrow{tm}^\eta,\overrightarrow{tm}\in D\vee\overrightarrow{tm}^\eta,\overrightarrow{tm}\notin D.\end{cases}$$

By the claim, $\Theta(\overrightarrow{tm})$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\eta>0$. As $\overrightarrow{tm}^1\subseteq\vec{t}\subseteq\overrightarrow{tm}$ and $R^{(1)}$ is distinguished in $\subseteq$ we get $\overrightarrow{tm}^1\ R^{(1)}\ \vec{t}$ and $U_{\vec{t}}\subseteq\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}$, by induction assumption. Therefore $\overline{U_{\vec{t}}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Theta}(\overrightarrow{tm})$, and $(\alpha^i_{t_i})_{i\in d}\in U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Theta}(\overrightarrow{tm})$ (even if $\eta=0$). Therefore $\overline{\Theta}$ admits a π-selector on $\mathcal{T}^{l+1}$. Indeed, we define, for each $i\in d$, $\overline{\theta}_i:\mathcal{X}\to\omega^\omega$ by $\overline{\theta}_i(t_im_i):=\alpha^i_{t_i}$ if $t_i\in\Pi''_iT_d$, 0^∞ otherwise.

- As T_d is a tree with $\varDelta^1_1$ suitable levels, we can apply Lemma 4.1.3. Thus Θ admits a π-selector θ on $\mathcal{T}^{l+1}$. We set, for $s\in\Pi_i[\mathcal{T}^{l+1}]$, $\alpha^i_s:=\theta_i(s)$.

- We choose $\varSigma^1_1$ sets $U_{\overrightarrow{tm}}$ with $d_{(\omega^\omega)^d}$-diameter at most 2^{-l-1} such that $\theta(\overrightarrow{tm})\in U_{\overrightarrow{tm}}\subseteq\Theta(\overrightarrow{tm})$ if $\overrightarrow{tm}\in\mathcal{T}^{l+1}$.

- Finally, we choose the O^i_{sq}'s. We first prove that $\alpha^i_{sq}\in O^i_s$ if $sq\in\Pi_i[\mathcal{T}^{l+1}]$, $i\in d$ and $i\leq l$.

Let $\overrightarrow{tm}\in\mathcal{T}^{l+1}$ such that $sq=t_im_i$. Then $\alpha^i_{sq}=\theta_i(sq)=\theta_i(t_im_i)$. As $\theta(\overrightarrow{tm})\in\Theta(\overrightarrow{tm})$ and $i\leq l$, we get $\alpha^i_{sq}\in O^i_{t_i}=O^i_s$.

Now we can define the O^i_{sq}'s. If $sq\in\Pi_i[\mathcal{T}^{l+1}]$, then we choose a Σ^1_1 set O^i_{sq}, with d_{ω^ω}-diameter at most 2^{-l-1}, such that

$$\alpha^i_{sq}\in O^i_{sq}\subseteq\begin{cases} O^i_s & \text{if } i\leq l,\\ \Omega_{\omega^\omega} & \text{otherwise.}\end{cases}$$

- This finishes the proof since $\vec{u}\ R^{(\rho)}\ \overrightarrow{tm}$ and $\vec{u}\neq\overrightarrow{tm}\ \Rightarrow\ \vec{u}\ R^{(\rho)}\ \overrightarrow{tm}^\rho\ R^{(\rho)}\ \overrightarrow{tm}$, by Lemma 4.3.2. $\square$

Now we come to the ambiguous classes.

Theorem 4.4.2 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ a successor ordinal, S^0,S^1 in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint, and A_0, A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*
(a) $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}=\emptyset$.
(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S^0,S^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 4.4.1. Assume that $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}\neq\emptyset$. We set $I^\varepsilon:=h[\lceil T_d\rceil\setminus S^\varepsilon]$, so that I^ε is a $\mathbf{\Pi}^0_\xi$ subset of $[\subseteq]$. We also set, for $\varepsilon\in 2$,

$$D^\varepsilon_1:=\big\{\vec{s}\in T_d\mid\vec{s}=\vec{\emptyset}\vee\exists\mathcal{B}\in\Pi^{-1}(I^\varepsilon)\ \ \vec{s}\in\mathcal{B}\big\},$$

and $D^\varepsilon_0:=T_d\setminus D^\varepsilon_1$. We set, for $\theta_0,\theta_1\in 2$, $D_{\theta_0,\theta_1}:=D^0_{\theta_0}\cap D^1_{\theta_1}$. For example, $\vec{\emptyset}\in D_{1,1}$.

- Conditions (4), (5), and (7) become the following:

(4) $U_{\vec{s}}\subseteq\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}$ if $\vec{s}\in D_{1,1}$,

(5) $U_{\vec{s}}\subseteq A_\varepsilon$ if $\vec{s}\in D_{\varepsilon,1-\varepsilon}$,

(7) $(\vec{s},\vec{t}\in D_{\varepsilon,1-\varepsilon}\wedge\vec{s}\ R^{(\eta)}\ \vec{t}\,)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$.

- Fix $\vec{\alpha}\in\lceil T_d\rceil$. There are $(\theta_0,\theta_1)\in 2^2$ and $k_0\in\omega$ such that, for $k\geq k_0$, $\vec{\alpha}|j_k\in D_{\theta_0,\theta_1}$. Thus $S^\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon)$.

- Let $(\alpha^i_\emptyset)_{i\in d}\in\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}\cap\Omega_{(\omega^\omega)^d}$, which is nonempty since $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}\neq\emptyset$ is Σ^1_1. We choose $U_{\vec{\emptyset}}$ with $(\alpha^i_\emptyset)_{i\in d}\in U_{\vec{\emptyset}}\subseteq\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}\cap\Omega_{(\omega^\omega)^d}$.

• The statement of the claim is now as follows:

Claim. Assume that $\eta>0$.

(a) $A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\eta\notin D_{\varepsilon,1-\varepsilon}$ and $\overrightarrow{tm}\in D_{\varepsilon,1-\varepsilon}$.

(b) $U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ otherwise.

The point is that $\overrightarrow{tm}^\eta\in D_{1,1}$ if $\overrightarrow{tm}^\eta\notin D_{\varepsilon,1-\varepsilon}$ since $\overrightarrow{tm}^\eta\in D_{\theta_0,\theta_1}$ with $\varepsilon\leq\theta_0$ and $1-\varepsilon\leq\theta_1$.

• In the same fashion, $\Theta(\overrightarrow{tm})$ is now defined as follows:

$$\Theta(\overrightarrow{tm}):=\left\{\begin{array}{l} A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{(\omega^\omega)^d}\text{ if }\overrightarrow{tm}^\eta\notin D_{\varepsilon,1-\varepsilon}\wedge\overrightarrow{tm}\in D_{\varepsilon,1-\varepsilon},\\ \\ U_{\overrightarrow{tm}^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\text{ otherwise.}\end{array}\right.$$

We conclude as in the proof of Theorem 4.4.1. □

Now we come to the limit case. We need some more definitions that can be found in [D-SR].

Definition 4.4.3 *(Debs-Saint Raymond) Let R be a tree relation on $c^{<\omega}$. If $t\in c^{<\omega}$, then $h_R(t)$ is the number of strict R-predecessors of t. So we have $h_R(t)=\mathit{Card}\big(P_R(t)\big)-1$.*

Let $\xi<\omega_1$ be an infinite limit ordinal. We say that a resolution family $(R^{(\rho)})_{\rho\leq\xi}$ is $uniform$ if

$$\forall k\in\omega\ \exists\eta_k<\xi\ \forall s,t\in c^{<\omega}\ \ \Big(\mathit{min}\big(h_{R^{(\xi)}}(s),h_{R^{(\xi)}}(t)\big)\leq k\ \wedge s\ R^{(\eta_k)}\ t\Big)\Rightarrow s\ R^{(\xi)}\ t.$$

We may (and will) assume that $\eta_k\geq 2$.

The following is part of Theorem I-6.6 in [D-SR].

Theorem 4.4.4 *(Debs-Saint Raymond) Let $\xi<\omega_1$ be an infinite limit ordinal, R a tree relation, $(I_n)_{n\in\omega}$ a sequence of $\mathbf{\Pi}^0_\xi$ subsets of $[R]$. Then there is a uniform resolution family $(R^{(\rho)})_{\rho\leq\xi}$ with*

(a) $R^{(0)}=R$.

(b) The canonical map $\Pi:[R^{(\xi)}]\to[R]$ is a continuous bijection.

(c) The set $\Pi^{-1}(I_n)$ is a closed subset of $[R^{(\xi)}]$ for each integer n.

Here again, the next result is essentially in [L7] (see Theorem 2.4.4).

Theorem 4.4.5 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ an infinite limit ordinal, S in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$, and A_0, A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*

(a) $\overline{A_0}^{\tau_\xi}\cap A_1=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 4.4.1.

• The set $I:=h[\lceil T_d\rceil\setminus S]$ is $\mathbf{\Pi}^0_\xi([\subseteq])$. Theorem 4.4.4 provides a uniform resolution family.

• If $\vec{t}\in c^{<\omega}$ then we set $\eta(\vec{t}\,):=\max\{\eta_{h_{R(\xi)}(\vec{s})+1}\mid\vec{s}\subseteq\vec{t}\,\}$. Note that $\eta(\vec{s}\,)\leq\eta(\vec{t}\,)$ if $\vec{s}\subseteq\vec{t}$.

• Conditions (6) and (7) become

(6) $(1\leq\rho\leq\eta(\vec{s}\,)\wedge\vec{s}\,R^{(\rho)}\,\vec{t}\,)\Rightarrow U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^{\tau_\rho}$,

(7) $\big((\vec{s},\vec{t}\in D\vee\vec{s},\vec{t}\notin D)\wedge\vec{s}\,R^{(\xi)}\,\vec{t}\,\big)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$.

Claim 1. Assume that $\overrightarrow{tm}^\rho\neq\overrightarrow{tm}^\xi$. Then $\rho+1\leq\eta(\overrightarrow{tm}^{\rho+1})$.

We argue by contradiction. We get $\rho+1>\rho\geq\eta(\overrightarrow{tm}^{\rho+1})\geq\eta_{h_{R(\xi)}(\overrightarrow{tm}^\xi)+1}=\eta_{h_{R(\xi)}(\overrightarrow{tm})}$. As $\overrightarrow{tm}^\rho\ R^{(\rho)}\ \overrightarrow{tm}$ we get $\overrightarrow{tm}^\rho\ R^{(\xi)}\ \overrightarrow{tm}$, and also $\overrightarrow{tm}^\rho=\overrightarrow{tm}^\xi$, which is absurd. $\diamond$

Note that $\xi_{n-1}<\xi_{n-1}+1\leq\eta(\overrightarrow{tm}^{\xi_{n-1}+1})\leq\eta(\overrightarrow{tm})$. This implies that $\overrightarrow{tm}^{\eta(\overrightarrow{tm})}=\overrightarrow{tm}^\xi$.

Claim 2. (a) The set $A_0\cap\bigcap_{1\leq\rho\leq\eta(\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\eta\in D$ and $\overrightarrow{tm}\notin D$.

(b) The set $U_{\overrightarrow{tm}^\xi}\cap\bigcap_{1\leq\rho<\eta(\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\xi,\overrightarrow{tm}\in D$ or $\overrightarrow{tm}^\xi,\overrightarrow{tm}\notin D$.

Indeed, we set $S_i:=U_{\overrightarrow{tm}^{\xi_i}}$, for $1\leq\xi_i\leq\xi$. By Claim 1 we can apply Lemma 4.2.3.(2).(c) and we are done. $\diamond$

• The map $\Theta:\mathcal{X}^d\to\Sigma^1_1\big((\omega^\omega)^d\big)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Theta(\overrightarrow{tm}):=\begin{cases} A_0\cap\bigcap_{1\leq\rho\leq\eta(\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{(\omega^\omega)^d}\text{ if }\overrightarrow{tm}^\eta\in D\wedge\overrightarrow{tm}\notin D,\\ \\ U_{\overrightarrow{tm}^\xi}\cap\bigcap_{1\leq\rho<\eta(\overrightarrow{tm})}\overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)\text{ if }\overrightarrow{tm}^\xi,\overrightarrow{tm}\in D\vee\overrightarrow{tm}^\xi,\overrightarrow{tm}\notin D.\end{cases}$$

We conclude as in the proof of Theorem 4.4.1, using the facts that $\eta_k\geq 1$ and $\eta(.)$ is increasing. $\square$

Now we come to the ambiguous classes.

Theorem 4.4.6 *Let T be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ an infinite limit ordinal, S^0,S^1 in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint, and A_0, A_1 disjoint Σ^1_1 subsets of $(\omega^\omega)^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*
(a) $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}=\emptyset$.
(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S^0,S^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proofs of Theorems 4.4.1, 4.4.2 and 4.4.5.

• The set $I^\varepsilon := h[\lceil T_d\rceil \setminus S^\varepsilon]$ is $\mathbf{\Pi}^0_\xi([\subseteq])$.

• The statement of Claim 2 is now as follows.

Claim 2. (a) $A_\varepsilon \cap \bigcap_{1\leq\rho\leq\eta(\overrightarrow{tm})} \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho} \cap (\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1} \cap (\Pi_{i\in d}\ X_i)$ if $\overrightarrow{tm}^\xi \notin D_{\varepsilon,1-\varepsilon}$ and $\overrightarrow{tm} \in D_{\varepsilon,1-\varepsilon}$.

(b) $U_{\overrightarrow{tm}^\xi} \cap \bigcap_{1\leq\rho<\eta(\overrightarrow{tm})} \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho} \cap (\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1}}^{\tau_1} \cap (\Pi_{i\in d}\ X_i)$ otherwise.

• In the same fashion, $\Theta(\overrightarrow{tm})$ is now defined as follows:

$$\Theta(\overrightarrow{tm}) := \begin{cases} A_\varepsilon \cap \bigcap_{1\leq\rho\leq\eta(\overrightarrow{tm})} \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho} \cap (\Pi_{i\in d}\ X_i) \cap \Omega_{(\omega^\omega)^d} \text{ if } \overrightarrow{tm}^\xi \notin D_{\varepsilon,1-\varepsilon} \wedge \overrightarrow{tm} \in D_{\varepsilon,1-\varepsilon}, \\ \\ U_{\overrightarrow{tm}^\xi} \cap \bigcap_{1\leq\rho<\eta(\overrightarrow{tm})} \overline{U_{\overrightarrow{tm}^\rho}}^{\tau_\rho} \cap (\Pi_{i\in d}\ X_i) \text{ otherwise.} \end{cases}$$

We conclude as in the proof of Theorem 4.4.5. □

Lemma 4.4.7 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets. Then the class of pot$(\mathbf{\Gamma})$ sets is closed under pre-images by products of continuous maps.*

Proof. Assume that $A \in \text{pot}(\mathbf{\Gamma})$, $A \subseteq \Pi_{i\in d}\ Y_i$, and $f_i : X_i \to Y_i$ is continuous. Let τ_i be a finer 0-dimensional Polish topology on Y_i such that $A \in \mathbf{\Gamma}\big(\Pi_{i\in d}\ (Y_i, \tau_i)\big)$. As $f_i : X_i \to (Y_i, \tau_i)$ is Borel, there is a finer 0-dimensional Polish topology σ_i on X_i such that $f_i : (X_i, \sigma_i) \to (Y_i, \tau_i)$ is continuous. Thus $(\Pi_{i\in d}\ f_i)^{-1}(A) \in \mathbf{\Gamma}\big(\Pi_{i\in d}\ (X_i, \sigma_i)\big)$ and $(\Pi_{i\in d}\ f_i)^{-1}(A) \in \text{pot}(\mathbf{\Gamma})$. □

Proof of Theorem 4.1 for ξ, assuming that Theorem 4.2.2 is proved for $\eta < \xi$.

(1) We assume that (a) does not hold. This implies that the X_i's are not empty.

- We first prove that we may assume that $X_i = \omega^\omega$ for each $i \in d$.

By 13.5 in [K], there is a finer zero-dimensional Polish topology τ_i on X_i, and, by 7.8 in [K], (X_i, τ_i) is homeomorphic to a closed subset F_i of ω^ω, via a map φ_i. By 2.8 in [K], there is a continuous retraction $r_i : \omega^\omega \to F_i$. Let A'_ε be the intersection of $\Pi_{i\in d}\ F_i$ with the pre-image of A_ε by $\Pi_{i\in d}\ (\varphi_i^{-1} \circ r_i)$. Then A'_0 and A'_1 are disjoint analytic subsets of $(\omega^\omega)^d$. Moreover, A'_0 is not separable from A'_1 by a pot$(\mathbf{\Pi}^0_\xi)$ set, since otherwise (a) would hold.

This gives $g_i : d^\omega \to \omega^\omega$ continuous with $S \subseteq (\Pi_{i\in d}\ g_i)^{-1}(A'_0)$ and $\lceil T_d\rceil \setminus S \subseteq (\Pi_{i\in d}\ g_i)^{-1}(A'_1)$. It remains to set $f_i(\alpha) := (\varphi_i^{-1} \circ r_i \circ g_i)(\alpha)$ if $\alpha \in d^\omega$.

- To simplify the notation, we may assume that T_d has Δ^1_1 levels, $\xi < \omega_1^{\text{CK}}$ and A_0, A_1 are $\Sigma^1_1\big((\omega^\omega)^d\big)$. Notice that $\overline{A_0}^{\tau_\xi} \cap A_1$ is not empty, since otherwise A_0 would be separable from A_1 by a set in $\mathbf{\Pi}^0_1(\tau_\xi) \subseteq \mathbf{\Pi}^0_\xi(\tau_1) \subseteq \text{pot}(\mathbf{\Pi}^0_\xi)$ set, which is absurd. So (b) holds, by Theorems 4.4.1 and 4.4.5 (as $\Pi''_i\lceil T_d\rceil$ is compact, we just have to compose with continuous retractions to get functions defined on d^ω). So (a) or (b) holds.

• If $P\in\mathrm{pot}(\mathbf{\Pi}^0_\xi)$ separates A_0 from A_1 and (b) holds, then $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(P)\subseteq\neg(\lceil T_d\rceil\setminus S)$. This implies that S is separable from $\lceil T_d\rceil\setminus S$ by a $\mathrm{pot}(\mathbf{\Pi}^0_\xi)$ set, by Lemma 4.4.7.

(2) We argue as in the proof of (1). Here we consider $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}$, and we apply Theorems 4.4.2 and 4.4.6. This finishes the proof. $\square$

Proof of Theorem 4.2.2. We assume that Theorem 4.1 is proved for ξ, and that Theorem 4.2.2 is proved for $\eta<\xi$.

(1) By Lemma 4.2.3, V_0 and $V_{<\xi}$ are $\mathit{\Pi}^1_1$.

(a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) are clear since $\mathit{\Delta}_{\omega^\omega}$ is Polish.

(b) $\Rightarrow$ (c) We argue by contradiction. As $\gamma\in\mathit{\Delta}^1_1$ we get $C_\gamma\in\mathit{\Delta}^1_1$. If $(\beta,\gamma)\in V_{<\xi}$, then $C_\gamma\in\mathrm{pot}(\mathbf{\Pi}^0_{<\xi})$, which is absurd. If $(\beta,\gamma)\in V_0$, then $C_\gamma\in\mathrm{pot}(\mathbf{\Pi}^0_0)\subseteq\mathrm{pot}(\mathbf{\Pi}^0_\xi)$, which is absurd. If $(\beta,\gamma)\notin V_{<\xi}\cup V_0$, then we get $\gamma'\in\mathit{\Delta}^1_1$ (see the definition of Φ before Theorem 4.2.2). As $\big((\beta)_n,(\gamma')_n\big)\in V_{<\xi}$, we get $C_{(\gamma')_n}\in\mathrm{pot}(\mathbf{\Pi}^0_{<\xi})$. Now the equality $\neg C_\gamma=\bigcup_{n\in\omega}\ C_{(\gamma')_n}$ implies that $C_\gamma\in\mathrm{pot}(\mathbf{\Pi}^0_\xi)$, which is absurd.

(d) $\Rightarrow$ (e) This comes from the proof of Theorem 4.1.(1).

(e) $\Rightarrow$ (f) This comes from Theorems 4.4.1 and 4.4.5.

(f) $\Rightarrow$ (a) This comes from Theorem 4.1.(1).

(c) $\Rightarrow$ (e) We argue by contradiction, so that $\overline{A_0}^{\tau_\xi}$ separates A_0 from A_1.

If $\xi=1$, then for each $\vec{\delta}\in A_1$ there is $(\tilde{\beta},\tilde{\gamma})\in(\mathit{\Delta}^1_1\times\mathit{\Delta}^1_1)\cap V_0$ such that $\vec{\delta}\in C_{\tilde{\gamma}}\subseteq\neg A_0$. The first reflection theorem gives $\beta,\gamma'\in\mathit{\Delta}^1_1$ such that $\big((\beta)_n,(\gamma')_n\big)\in V_0$ for each integer n and $A_1\subseteq U:=\bigcup_{n\in\omega}\ C_{(\gamma')_n}\subseteq\neg A_0$. We choose $\gamma\in\mathit{\Delta}^1_1\cap W$ with $\neg C_\gamma=U$, and (β,γ) contradicts (c).

If $\xi\geq 2$, then by induction assumption and the first reflection theorem there are $\beta,\gamma'\in\mathit{\Delta}^1_1$ with $\big((\beta)_n,(\gamma')_n\big)\in V_{<\xi}$ and $C_{(\gamma')_n}\subseteq\neg A_0$, for each integer n, and $A_1\subseteq U:=\bigcup_n\ C_{(\gamma')_n}$. But U is $\mathit{\Delta}^1_1\cap\mathrm{pot}(\mathbf{\Sigma}^0_\xi)$ and separates A_1 from A_0. So let $\gamma\in\mathit{\Delta}^1_1\cap W$ with $\neg C_\gamma=U$. We have $(\beta,\gamma)\in V_\xi$ and C_γ separates A_0 from A_1, which is absurd.

(2) It is clear that V_ξ is $\mathit{\Pi}^1_1$.

(3) We argue as in the proof of (1), except for the implication (c) $\Rightarrow$ (e) (for the implication (e) $\Rightarrow$ (f) we use Theorems 4.4.2 and 4.4.6).

(c) $\Rightarrow$ (e) We argue by contradiction. By 4D.2 in [M], there are $W\in\mathit{\Pi}^1_1(\omega)$ and a partial function $\mathbf{d}:\omega\to\omega^\omega$, $\mathit{\Pi}^1_1$-recursive on W, such that $\mathbf{d}''W$ is the set of $\mathit{\Delta}^1_1$ points of ω^ω. We define

$$\Pi_{A_\varepsilon}:=\big\{n\in\omega\mid(n)_0,(n)_1\in W\wedge\big(\mathbf{d}\big((n)_0\big),\mathbf{d}\big((n)_1\big)\big)\in V_{<\xi}\wedge C_{\mathbf{d}((n)_1)}\cap A_\varepsilon=\emptyset\big\}.$$

Then $\Pi_{A_\varepsilon}\in\mathit{\Pi}^1_1$ and $\forall\vec{\beta}\in(\omega^\omega)^d\ \ \exists n\in\Pi_{A_0}\cup\Pi_{A_1}\ \ \vec{\beta}\in C_{\mathbf{d}((n)_1)}$ since $\overline{A_0}^{\tau_\xi}\cap\overline{A_1}^{\tau_\xi}=\emptyset$ (we use the induction assumption). By the first reflection theorem there is $D\in\mathit{\Delta}^1_1(\omega)$ such that $D\subseteq\Pi_{A_0}\cup\Pi_{A_1}$ and $\forall\vec{\beta}\in(\omega^\omega)^d\ \ \exists n\in D\ \ \vec{\beta}\in C_{\mathbf{d}((n)_1)}$.

As $\mathit{\Pi}^1_1$ has the reduction property, we can find $\Pi'_{A_\varepsilon}\in\mathit{\Pi}^1_1$ disjoint such that $\Pi'_{A_\varepsilon}\subseteq\Pi_{A_\varepsilon}$ and $\Pi'_{A_0}\cup\Pi'_{A_1}=\Pi_{A_0}\cup\Pi_{A_1}$. We set $\Delta:=\bigcup_{n\in D\cap\Pi'_{A_1}}\ C_{\mathbf{d}((n)_1)}\setminus(\bigcup_{q<n}\ C_{\mathbf{d}((q)_1)})$. Then

$$\neg\Delta=\bigcup_{n\in D\cap\Pi'_{A_0}}\ C^{(\omega^\omega)^d}_{\mathbf{d}((n)_1)}\setminus(\bigcup_{q<n}\ C^{(\omega^\omega)^d}_{\mathbf{d}((q)_1)}),$$

which proves that $\Delta\in\mathit{\Delta}^1_1\cap\text{pot}(\mathbf{\Delta}^0_\xi)$, and separates A_0 from A_1. Let $(\beta,\gamma),(\beta',\gamma')\in(\mathit{\Delta}^1_1\times\mathit{\Delta}^1_1)\cap V_\xi$ with $\Delta=C_\gamma$ and $\neg\Delta=C_{\gamma'}$. Then we get a contradiction with (c). $\square$

Remarks. The assertions 4.2.3.(2).(a) and 4.2.3.(2).(b) admit uniform versions in the following sense. By 3E.2, 3F.6 and 3H.1 in [M], there is $S:\omega^\omega\times\omega^\omega\rightarrow\omega^\omega$ recursive such that for each recursively presented Polish space X there is a universal set $\mathcal{U}^X\in\mathit{\Pi}^1_1\big((\omega^\omega)^d\big)$ satisfying the following properties:
- $\mathbf{\Pi}^1_1(X)=\{\mathcal{U}^X_\alpha\mid\alpha\in\omega^\omega\}$,
- $\mathit{\Pi}^1_1(X)=\{\mathcal{U}^X_\alpha\mid\alpha\in\omega^\omega$ recursive$\}$,
- $(\alpha,\beta,x)\in\mathcal{U}^{\omega^\omega\times X}\Leftrightarrow\big(S(\alpha,\beta),x\big)\in\mathcal{U}^X$.

We set $\mathcal{U}:=\mathcal{U}^{(\omega^\omega)^d}$. The following relations are $\mathit{\Pi}^1_1$:

$$\begin{aligned}&Q(\alpha,\beta,\gamma)\Leftrightarrow\alpha\in\text{WO}\wedge(\beta,\gamma)\in V_{|\alpha|},\\&R(\alpha,\beta,\vec\delta)\Leftrightarrow\alpha\in\mathit{\Delta}^1_1\cap\text{WO}\wedge|\alpha|\geq 1\wedge\vec\delta\notin\overline{\neg\mathcal{U}_\beta}^{\tau_{|\alpha|}}.\end{aligned}$$

Indeed, this comes from the proof of Lemma 4.2.3.

• One can give simpler examples $\mathbb{S}^0,\mathbb{S}^1$ for which Corollary 4.2 is fullfilled when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$. Indeed, recall the map b_ω defined before Lemma 2.3. As $|b_\omega(n)|\leq n$ for each integer n, we can define the sequence $s^\omega_n:=b_\omega(n)0^{n-|b_\omega(n)|}$. We set $\mathbb{S}^1:=\overline{\mathbb{S}^0}\setminus\mathbb{S}^0$, where

$$\mathbb{S}^0:=\Big\{\big(0s^\omega_n0\gamma,...,0s^\omega_nn\gamma,(n+1)s^\omega_n(n+1)\gamma,(n+1)s^\omega_n(n+2)\gamma,...\big)\mid(n,\gamma)\in\omega\times\omega^\omega\Big\}$$

(we do not really need T_ω when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$). We have $\mathbb{S}^0=(\Pi_{i\in d}\ f_i)^{-1}(A_0)\cap\overline{\mathbb{S}^0}$ if (b) holds. Let us denote this by $\mathbb{S}^0\leq A_0$ (we have a quasi-order, by continuity of the f_i's).

• The fact that T_d has finite levels was used to give a proof of Corollary 4.2 as simple as possible. The tree T_d has finite levels when $d<\omega$, and not always when $d=\omega$. This is one of the main new points in the case of the infinite dimension. Let us specify this.

(a) We saw in the proof of Proposition 2.2 that the tree $\tilde T_d$ generated by an effective frame is a tree with one-sided almost acyclic levels. As before Lemma 2.6, we can define

$$\tilde S^\omega_{C_1}:=\{\vec\alpha\in\lceil\tilde T_d\rceil\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\in C_1\},$$

which is not separable from $\lceil\tilde T_d\rceil\setminus\tilde S^\omega_{C_1}$ by a potentially closed set, since otherwise $S^\omega_{C_1}$ would be separable from $\lceil T_d\rceil\setminus S^\omega_{C_1}$ by a potentially closed set, which would contradict Lemmas 2.6 and 3.4.

But $\mathbb{A}_0:=\{0^{1+n}(1+n)^\infty\mid n\in\omega\}\subseteq\omega^\omega$ is not potentially closed since $0^\infty\in\overline{\mathbb{A}_0}\setminus\mathbb{A}_0$ and the topology on ω is discrete. And one can prove, in a straightforward way, that $\tilde S^\omega_{C_1}\not\leq\mathbb{A}_0$ and $\mathbb{A}_0\not\leq\tilde S^\omega_{C_1}$. This proves that the finiteness of the levels of T_d is useful. But we will see that it is not necessary.

(b) We define $o\!:\!\{s\!\in\!2^{<\omega}\mid 0\!\not\subseteq\! s\}\!\to\!\omega^{<\omega}$ such that $|o(s)|\!=\!|s|$ by

$$o(10^{n_0}10^{n_1}...10^{n_l}):=0^{1+n_0}(1+n_0)^{1+n_1}...\big((1+n_0)+...+(1+n_{l-1})\big)^{1+n_l}.$$

In other words, we have $o(s)(i)=i$ if $s(i)=1$, $o(s)(i)=o(s)(i-1)$ if $s(i)=0$. Note that o is an injective homomorphism, in the sense that $o(s)\subseteq o(t)$ if $s\subseteq t$. This implies that we can extend o to a continuous map from the basic clopen set N_1 into ω^ω by the formula $o(\alpha):=\sup_{m\in\omega} o(\alpha|m)$.

We set $F_\omega := \big\{(m_i\alpha_i)_{i\in\omega}\in(\omega^\omega)^\omega \mid \vec\alpha\in\lceil\tilde T_\omega\rceil$ and $\forall i\in\omega\ \ m_i=o(\alpha_0\Delta\alpha_1)(i)\big\}$, and we put $\underline{S}^\omega_{C_\xi}:=\{(m_i\alpha_i)_{i\in\omega}\in F_\omega \mid \mathcal{S}(\alpha_0\Delta\alpha_1)\in C_\xi\}$. One can take $\mathbb{S}^\omega_\xi=\underline{S}^\omega_{C_\xi}$, and the proof is much more complicated than the one we gave. But the tree associated with $\overline{\underline{S}^\omega_{C_\xi}}=F_\omega$ is

$$\big\{\vec\emptyset\big\}\cup\big\{(m_is_i)_{i\in\omega}\in(\omega^\omega)^{<\omega}\mid (m_i)_{i\in\omega}\in o''[N_1]\ \text{ and }\ \vec s\in\tilde T_\omega\ \text{ and }\ \forall i<|\vec s\,|\ \ m_i=o(s_0\Delta s_1)(i)\big\},$$

and has infinite levels. This proves that the finiteness of the levels of the tree associated with $\overline{\mathbb{S}^\omega_\xi}$ is not necessary.

(c) In [L8], an extension to any dimension of the Kechris-Solecki-Todorčević dichotomy about analytic graphs is proved. In [L5], it is proved that Corollary 4.2 is a consequence of the Kechris-Solecki-Todorčević dichotomy when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$. This works as well when $d<\omega$, but not when $d=\omega$. More specifically, let $\mathbb{G}:=\{\alpha\in\omega^\omega\mid\forall m\in\omega\ \exists n\geq m\ \ s^\omega_n0\subseteq\alpha\}$ and

$$\mathbb{A}_\omega:=\{(s^\omega_n i\gamma)_{i\in\omega}\mid n\in\omega\wedge\gamma\in\omega^\omega\}.$$

Then the extension to the case where $d=\omega$ of the Kechris-Solecki-Todorčević dichotomy works with $\mathbb{G}^\omega\cap\mathbb{A}_\omega$ (see [L8]). But one can prove the following result:

Theorem 4.4.8 *Let X be a recursively presented Polish space, σ_X the topology on X^ω generated by $\{\Pi_{i\in\omega}\, C_i\mid C\in\Delta^1_1(\omega\times X)\}$, and A a Δ^1_1 subset of X^ω. Then exactly one of the following holds:*
(a) $\overline{A}^{\sigma_X}\setminus A=\emptyset$.
(b) $\mathbb{G}^\omega\cap\mathbb{A}_\omega\leq A$.

In particular, $\mathbb{G}^\omega\cap\mathbb{A}_\omega\not\leq\mathbb{A}_0$ and we cannot take $\mathbb{S}^\omega_1=\mathbb{G}^\omega\cap\mathbb{A}_\omega$.

5 The proof of Theorem 1.7

5.1 Some material in dimension one

The material in this subsection is due to A. Louveau and J. Saint Raymond, and can be found in [Lo-SR1] or [Lo-SR2]. However, some changes are needed for our purposes, and moreover some proofs are left to the reader in these papers. So we will sometimes give some proofs. The following definition can be found in [Lo-SR2] (see Definition 1.5).

Definition 5.1.1 *Let* $1 \leq \xi < \omega_1$, $\mathbf{\Gamma}$ *and* $\mathbf{\Gamma}'$ *two classes. Then*

$$A \in S_\xi(\mathbf{\Gamma}, \mathbf{\Gamma}') \;\Leftrightarrow\; A = \bigcup_{p\geq 1} (A_p \cap C_p) \cup \left(B \setminus \bigcup_{p\geq 1} C_p \right)$$

for some sequence of sets A_p *in* $\mathbf{\Gamma}$, $B \in \mathbf{\Gamma}'$, *and a sequence* $(C_p)_{p\geq 1}$ *of pairwise disjoint* $\mathbf{\Sigma}^0_\xi$ *sets.*

Now we come to the definition of $second\ type\ descriptions$ of non self-dual Wadge classes of Borel sets, which are elements of ω_1^ω, sometimes identified with $(\omega_1^\omega)^\omega$. This definition can also be found in [Lo-SR2] (see Definition 1.6).

Definition 5.1.2 *The relations "u is a $second\ type\ description$" and "$u\ describes$ $\mathbf{\Gamma}$" (written* $u \in \mathcal{D}$ *and* $\mathbf{\Gamma}_u = \mathbf{\Gamma}$ *- ambiguously) are the least relations satisfying*
(a) If $u = 0^\infty$, *then* $u \in \mathcal{D}$ *and* $\mathbf{\Gamma}_u = \{\emptyset\}$.
(b) If $u = \xi^\frown 1^\frown u^*$, *with* $u^* \in \mathcal{D}$ *and* $u^*(0) = \xi$, *then* $u \in \mathcal{D}$ *and* $\mathbf{\Gamma}_u = \check{\mathbf{\Gamma}}_{u^*}$.
(c) If $u = \xi^\frown 2^\frown < u_p >$ *satisfies* $\xi \geq 1$, $u_p \in \mathcal{D}$, *and* $u_p(0) \geq \xi$ *or* $u_p(0) = 0$, *then* $u \in \mathcal{D}$ *and* $\mathbf{\Gamma}_u = S_\xi(\bigcup_{p\geq 1} \mathbf{\Gamma}_{u_p}, \mathbf{\Gamma}_{u_0})$.

Remark. If $A \in S_\xi(\bigcup_{p\geq 1} \mathbf{\Gamma}_{u_p}, \mathbf{\Gamma}_{u_0})$, then A has a decomposition as in Definition 5.1.1, and A_p is in $\bigcup_{p\geq 1} \mathbf{\Gamma}_{u_p}$. But we may assume that $A_p \in \mathbf{\Gamma}_{u_{(p)_0+1}}$, using the fact that C_p may be empty if necessary. This remark will be useful in the sequel, since it specifies the class of A_p.

The following result can be found in [Lo-SR2] (see Section 3).

Theorem 5.1.3 *Let* $\mathbf{\Gamma}$ *be a non self-dual Wadge class of Borel sets. Then there is* $u \in \mathcal{D}$ *such that* $\mathbf{\Gamma}(\omega^\omega) = \mathbf{\Gamma}_u(\omega^\omega)$. *Conversely,*

$$\mathbf{\Gamma}_u := \{ f^{-1}(A) \mid f : X \to \omega^\omega \text{ continuous } \wedge X \text{ 0-dimensional Polish space } \wedge A \in \mathbf{\Gamma}_u(\omega^\omega) \}$$

is a non self-dual Wadge class of Borel sets if $u \in \mathcal{D}$.

If $\eta \leq \xi < \omega_1$, then $\xi - \eta$ is the unique ordinal θ with $\eta + \theta = \xi$. The following definition can be found in [Lo-SR2] (see Definition 1.9).

Definition 5.1.4 *Let* $\eta < \omega_1$ *and* $u \in \mathcal{D}$. *We define* $u^\eta \in \mathcal{D}$ *as follows:*
(a) If $u(0) = 0$, *then* $u^\eta := u$.
(b) If $u = \xi 1 u^*$, *with* $\xi \geq 1$, *then* $u^\eta := \big(1 + \eta + (\xi - 1)\big) 1 (u^*)^\eta$.
(c) If $u = \xi 2 < u_p >$, *with* $\xi \geq 1$, *then* $u^\eta := \big(1 + \eta + (\xi - 1)\big) 2 < (u_p)^\eta >$.

The following result can be found in [Lo-SR2] (see Proposition 1.10).

Proposition 5.1.5 *(a) If* $f : \omega^\omega \to \omega^\omega$ *is* $\mathbf{\Sigma}^0_{1+\eta}$*-measurable, and* $A \in \mathbf{\Gamma}_u(\omega^\omega)$ *for some* $u \in \mathcal{D}$, *then* $f^{-1}(A) \in \mathbf{\Gamma}_{u^\eta}$.
(b) The set $\mathcal{D}$ *is the least subset* $D \subseteq \mathcal{D}$ *such that* $0^\infty \in D$, $u(0)1u \in D$ *if* $u \in D$, $12 < u_p > \in D$ *if* $u_p \in D$ *for each* $p \in \omega$, *and* $u^\eta \in D$ *if* $u \in D$, *for each* $\eta < \omega_1$.

Recall the definition of an independent η-function (see Definition 3.3).

Example. Let $\tau:\omega\rightarrow\omega$ be one-to-one (in [Lo-SR2] just before Lemma 2.5, increasing maps are considered, but here we relax this condition). We define $\tilde{\tau}:2^\omega\rightarrow 2^\omega$ by $\tilde{\tau}(\alpha):=\alpha\circ\tau$. Clearly $\tilde{\tau}$ is an independent 0-function, with $\pi(k)=\tau^{-1}(k)$ if k is in the range of τ, 0 otherwise. We now describe an important instance of this situation.

Example. Let n be an integer, and $\mathcal{S}$ the shift map (see the notation before Definition 2.5). Then $\mathcal{S}^n$ is an independent 0-function. Indeed, if we set $\tau^n(m):=m+n$, then $\mathcal{S}^n=\tilde{\tau^n}$, by induction on n. In particular, $\mathrm{Id}_{2^\omega}=\mathcal{S}^0$ is an independent 0-function.

The next result is essentially Lemma 2.5 in [Lo-SR2], which is given without proof, so we give the details here.

Lemma 5.1.6 *Let $\tau:\omega\rightarrow\omega$ be one-to-one, ρ an independent η-function. Then $\tilde{\tau}\circ\rho$ is an independent η-function.*

Proof. Let π associated with ρ. We define $\pi':\omega\rightarrow\omega$ by $\pi'(k):=\tau^{-1}\big(\pi(k)\big)$ if $\pi(k)$ is in the range of τ, 0 otherwise, so that $\pi'(k)=m$ if $\pi(k)=\tau(m)$. If m is an integer, then $(\tilde{\tau}\circ\rho)(\alpha)(m)=\rho(\alpha)\big(\tau(m)\big)$ depends only of the values of α on $\pi^{-1}\big(\{\tau(m)\}\big)\subseteq(\pi')^{-1}(\{m\})$.

If $\xi=0$ (resp., $\xi=\theta+1$, $\xi=\sup_{m\in\omega}\ \theta_m$), then $\mathcal{C}_m=\{\alpha\in 2^\omega\mid\rho(\alpha)\big(\tau(m)\big)=1\}$ is $\mathbf{\Delta}^0_1$-complete (resp., $\mathbf{\Pi}^0_{1+\theta}$-strategically complete, $\mathbf{\Pi}^0_{1+\theta_{\tau(m)}}$-strategically complete). We are done since $\xi=\sup_{p\geq 1}\ \theta_{\tau(m_p)}$ if ξ is a limit ordinal (τ is one-to-one). $\square$

After Definition 3.3, we saw that ρ^η_0 is an independent η-function. We will actually prove more, actually a result which is essentially Theorem 2.4.(b) in [Lo-SR2].

Theorem 5.1.7 *Let $\eta,\xi<\omega_1$, ρ an independent ξ-function. Then $\rho^\eta_0\circ\rho$ is an independent $(\xi+\eta)$-function.*

Proof. Note first that if $\varepsilon\in 2$, $\rho^\varepsilon:2^\omega\rightarrow 2^\omega$ is equipped with π^ε such that $\rho^\varepsilon(\alpha)(m)$ depends only on the values of α on $(\pi^\varepsilon)^{-1}(\{m\})$, then $(\rho^0\circ\rho^1)(\alpha)(m)$ depends only on the values of $\rho^1(\alpha)$ on $(\pi^0)^{-1}(\{m\})$, so it depends only on the values of α on $(\pi^1)^{-1}\big((\pi^0)^{-1}(\{m\})\big)$, so that if we set $\pi:=\pi^0\circ\pi^1$, then $(\rho^0\circ\rho^1)(\alpha)(m)$ depends only on the values of α on $\pi^{-1}(\{m\})$.

• We argue by induction on η. The result is clear for $\eta=0$. So assume that $\eta=\theta+1$, so that $\rho^\eta_0\circ\rho=\rho_0\circ\rho^\theta_0\circ\rho$. The induction assumption implies that $\rho^\theta\circ\rho$ is an independent $(\xi+\theta)$-function. The fact that ρ_0 is an independent 1-function and the previous point prove the existence of π_η such that $(\rho^\eta_0\circ\rho)(\alpha)(m)$ depends only on the values of α on $\pi^{-1}_\eta(\{m\})$.

We set $A_n:=\{\alpha\in 2^\omega\mid(\rho^\theta_0\circ\rho)(\alpha)(<m,n>)=1\}$. Let us prove that $\bigcap_{n\in\omega}\ \neg A_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Assume first that $\xi+\theta\neq 0$. As $\rho^\theta\circ\rho$ is an independent $(\xi+\theta)$-function, A_n is $\mathbf{\Pi}^0_{1+\theta_n}$-strategically complete, for some $\theta_n<\xi+\theta$ satisfying $\theta_n+1=\xi+\theta$ if $\xi+\theta$ is a successor ordinal, $\sup_{n\in\omega}\ \theta_n=\xi+\theta$ if $\xi+\theta$ is a limit ordinal. Note that $\xi+\theta=\sup_{n\in\omega}\ (\theta_n+1)$. As $\rho^\theta\circ\rho$ is an independent $(\xi+\theta)$-function, there is π_θ such that $(\rho^\theta_0\circ\rho)(\alpha)(q)$ depends only on the values of α on $\pi^{-1}_\theta(\{q\})$. We set $\pi(\alpha)(k):=\big(\pi_\theta(\alpha)\big)_1$, so that the fact that $\alpha\in A_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. By Lemma 3.7 in [Lo-SR1], $\bigcap_{n\in\omega}\ \neg A_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Assume now that $\xi+\theta=0$. Then $A_n := \{\alpha\in 2^\omega \mid \rho(\alpha)(< m,n >)=1\}$ is $\mathbf{\Delta}^0_1$-complete since ρ is an independent 0-function. Let B be a closed subset of ω^ω, $(B_n)_{n\in\omega}$ a sequence of clopen subsets with $B=\bigcap_{n\in\omega}\ B_n$, and $g_n:\omega^\omega\to 2^\omega$ continuous with $B_n=g_n^{-1}(\neg A_n)$. As ρ is an independent 0-function, there is π_ρ such that $\rho(\alpha)(q)$ depends only on the values of α on $\pi_\rho^{-1}(\{q\})$. We set $\pi(\alpha)(k) := \big(\pi_\rho(\alpha)\big)_1$, so that the fact that $\alpha\in A_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. We define $g:\omega^\omega\to 2^\omega$ by $g(\beta)(k):=g_{\pi(k)}(\beta)(k)$, so that g is continuous. Moreover, $\beta\in B_n \Leftrightarrow g_n(\beta)\notin A_n \Leftrightarrow g(\beta)\notin A_n$ since the fact that $\alpha\in A_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. Thus $B=g^{-1}(\bigcap_{n\in\omega}\ \neg A_n)$ and $\bigcap_{n\in\omega}\ \neg A_n$ is $\mathbf{\Pi}^0_1$-complete. Therefore $\bigcap_{n\in\omega}\ \neg A_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Now note that

$$\bigcap_{n\in\omega}\ \neg A_n = \{\alpha\in 2^\omega \mid \forall n\in\omega\ \ (\rho^\theta_0\circ\rho)(\alpha)(< m,n >)=0\}$$

$$= \{\alpha\in 2^\omega \mid (\rho_0\circ\rho^\theta_0\circ\rho)(\alpha)(m)=1\} = \{\alpha\in 2^\omega \mid (\rho^\eta_0\circ\rho)(\alpha)(m)=1\}.$$

Thus $\{\alpha\in 2^\omega \mid (\rho^\eta_0\circ\rho)(\alpha)(m)=1\}$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete for each m, and $\xi+\eta=\xi+\theta+1$, so that $\rho^\eta_0\circ\rho$ is an independent $(\xi+\eta)$-function.

• Assume now that η is a limit ordinal. In the definition of ρ^η_0 we fixed a sequence $(\theta^\eta_m)_{m\in\omega}\subseteq\eta$ of successor ordinals with $\Sigma_{m\in\omega}\ \theta^\eta_m=\eta$. As $\rho_0^{\theta^\eta_m}$ is an independent θ^η_m-function, we get $\pi^\eta_m:\omega\to\omega$. We define $\pi_{m,m+1}:\omega\to\omega$ by $\pi_{m,m+1}(k):=k$ if $k<m$, $\pi^\eta_m(k-m)+m$ if $k\geq m$. Let us check that $\rho_0^{(m,m+1)}(\alpha)(i)$ depends only on the values of α on $\pi^{-1}_{m,m+1}(\{i\})$. It is clearly the case if $i<m$. So assume that $i\geq m$. Note that $\pi_{m,m+1}(k)=i$ if $k\in(\pi^\eta_m)^{-1}(\{i-m\})+m$, and we are done. Now the first point of this proof gives $\pi_{0,m+1}:\omega\to\omega$ such that $\rho_0^{(0,m+1)}(\alpha)(i)$ depends only on the values of α on $\pi^{-1}_{0,m+1}(\{i\})$. We will check that $\rho^\eta_0(\alpha)(m):=\rho_0^{(0,m+1)}(\alpha)(m)$ depends only on the values of α on $E_m:=\pi^{-1}_{0,m+1}(\{m\})\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1))$. We actually prove something stronger: for each integer k, $\rho_0^{(0,m+1)}(\alpha)(k+m)$ depends only on the values of α on

$$\pi^{-1}_{0,m+1}(\{k+m\})\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1)).$$

We argue by induction on m. For $m=0$, the result is clear. Assume that the result is true for m. Note that $\rho_0^{(0,m+2)}(\alpha)(k+m+1)$ depends only on the values of α on $\pi^{-1}_{0,m+2}(\{k+m+1\})$. But

$$\rho_0^{(0,m+2)}(\alpha)(k+m+1)=\rho_0^{(m+1,m+2)}\big(\rho_0^{(0,m+1)}(\alpha)\big)(k+m+1)=\rho_0^{\theta^\eta_{m+1}}\Big(\mathcal{S}^{m+1}\big(\rho_0^{(0,m+1)}(\alpha)\big)\Big)(k),$$

and we are done since $\rho_0^{(0,m+2)}(\alpha)(\alpha)(k+m+1)$ depends only on the values of $\mathcal{S}^{m+1}\big(\rho_0^{(0,m+1)}(\alpha)\big)$, which depends only on the values of α on $\pi^{-1}_{0,m+1}(\neg(m+1))\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1))$.

As the E_m's are pairwise disjoint, we can define a map $\pi^\eta:\omega\to\omega$ by $\pi^\eta(k):=m$ if $k\in E_m$, and 0 if $k\notin\bigcup_{m\in\omega}\ E_m$. Now it is clear that $\rho^\eta_0(\alpha)(m)$ depends only on the values of α on $(\pi^\eta)^{-1}(\{m\})$. The first point of this proof gives $\pi_\eta:\omega\to\omega$ such that $(\rho^\eta_0\circ\rho)(\alpha)(m)$ depends only on the values of α on $\pi^{-1}_\eta(\{m\})$.

Let ζ^η_m such that $\theta^\eta_m := \zeta^\eta_m + 1$, and $\theta_m := \xi + \Sigma_{l<m}\ \theta^\eta_l + \zeta^\eta_m$, so that $\theta_m < \xi + \eta$ and $\sup_{p\geq 1}\ \theta_{m_p} = \xi + \eta$ for each one-to-one sequence $(m_p)_{p\geq 1}$ of integers. It remains to see that

$$\mathcal{C}_m := \{\alpha \in 2^\omega \mid (\rho^\eta_0 \circ \rho)(\alpha)(m) = 1\}$$

is $\mathbf{\Pi}^0_{1+\theta_m}$-strategically complete for each integer m.

Let us check that $\mathcal{S}^m \circ \rho_0^{(0,m+1)} = \rho_0^{\theta^\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S} \circ \rho_0^{\theta^\eta_{m-l-1}})$ for each integer m. We argue by induction on m. For $m = 0$ the property is clear since $\rho_0^{(0,1)} = \rho_0^{\theta^\eta_0}$. Assume that the property is true for m. Then

$$\mathcal{S}^{m+1} \circ \rho_0^{(0,m+2)} = \rho_0^{\theta^\eta_{m+1}} \circ \mathcal{S}^{m+1} \circ \rho_0^{(0,m+1)} = \rho_0^{\theta^\eta_{m+1}} \circ \mathcal{S} \circ \mathcal{S}^m \circ \rho_0^{(0,m+1)}$$

$$= \rho_0^{\theta^\eta_{m+1}} \circ \mathcal{S} \circ \rho_0^{\theta^\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S} \circ \rho_0^{\theta^\eta_{m-l-1}}) = \rho_0^{\theta^\eta_{m+1}} \circ\ \circ_{l\leq m}\ (\mathcal{S} \circ \rho_0^{\theta^\eta_{m-l}})$$

since in the last induction we proved that $\mathcal{S}^{m+1} \circ \rho_0^{(0,m+2)} = \rho_0^{\theta^\eta_{m+1}} \circ \mathcal{S}^{m+1} \circ \rho_0^{(0,m+1)}$. Thus

$$\mathcal{C}_m = \{\alpha \in 2^\omega \mid \rho_0^{(0,m+1)}\big(\rho(\alpha)\big)(m) = 1\} = \{\alpha \in 2^\omega \mid (\mathcal{S}^m \circ \rho_0^{(0,m+1)} \circ \rho)(\alpha)(0) = 1\}$$

$$= \{\alpha \in 2^\omega \mid \big(\rho_0^{\theta^\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S} \circ \rho_0^{\theta^\eta_{m-l-1}}) \circ \rho\big)(\alpha)(0) = 1\}.$$

So it is enough to see that $\rho^m := \rho_0^{\theta^\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S} \circ \rho_0^{\theta^\eta_{m-l-1}}) \circ \rho$ is an independent $(\theta_m + 1)$-function.

We argue by induction on m. For $m = 0$, we are done since $\rho_0^{\theta^\eta_0} \circ \rho$ is by induction assumption an independent $(\xi + \theta^\eta_0)$-function, and $\xi + \theta^\eta_0 = \xi + \zeta^\eta_0 + 1 = \theta_0 + 1$. Assume that the property is true for m. Then $\rho^{m+1} = \rho_0^{\theta^\eta_{m+1}} \circ \mathcal{S} \circ \rho^m$. By induction assumption, ρ^m is an independent $(\theta_m + 1)$-function. By Lemma 5.1.6 and the example just before it, $\mathcal{S} \circ \rho^m$ is also an independent $(\theta_m + 1)$-function. By induction assumption, ρ^{m+1} is an independent $(\theta_m + 1 + \theta^\eta_{m+1})$-function, and

$$\theta_m + 1 + \theta^\eta_{m+1} = \xi + \Sigma_{l<m}\ \theta^\eta_l + \zeta^\eta_m + 1 + \theta^\eta_{m+1} = \xi + \Sigma_{l\leq m}\ \theta^\eta_l + \zeta^\eta_{m+1} + 1 = \theta_{m+1} + 1.$$

This finishes the proof. □

5.2 Some complicated sets

Now we come to the existence of complicated sets, as in the statement of Theorem 1.7. Their construction is based on Theorem 2.7 in [Lo-SR2] that we now change. The main problem is that we want to ensure the ccs conditions of Lemma 2.6. To do this, we modify the definition of the maps τ_i of Lemma 2.11 in [Lo-SR2].

Notation. Let i be an integer. We define $\tau_i : \omega \to \omega$ by

$$\tau_i(k) := \begin{cases} < 0, k > \text{ if } i = 0, \\ \\ << i, (k)_0 >, (k)_1 > \text{ if } i \geq 1, \end{cases}$$

so that τ_i is one-to-one. This allows us to define, for each $\alpha \in 2^\omega$, $\alpha_i := \tilde{\tau}_i(\alpha)$. If $s \in (\omega \setminus \{0\})^{<\omega}$, then we set $\tilde{\tau}_s := \tilde{\tau}_{s(0)} \circ ... \circ \tilde{\tau}_{s(|s|-1)}$.

Lemma 5.2.1 *Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets, and H a $\mathbf{\Gamma}$-strategically complete set. Then*

(a) The set $\tilde{\tau_i}^{-1}(H)$ is $\mathbf{\Gamma}$-strategically complete for each integer i.

(b) Assume that $\tau:\omega\rightarrow\omega$ is one-to-one such that the fact that $\alpha\in H$ depends only on $\alpha\circ\tau$. Then $L:=\{\alpha\circ\tau\mid\alpha\in H\}$ is $\mathbf{\Gamma}$-strategically complete.

Proof. (a) As $\tilde{\tau_i}$ is continuous, $\tilde{\tau_i}^{-1}(H)\in\mathbf{\Gamma}(2^\omega)$. We define a continuous map $f_{\tau_i}:2^\omega\rightarrow 2^\omega$ by $f_{\tau_i}(\alpha)(m):=\alpha\big(\tau_i^{-1}(m)\big)$ if m is in the range of τ_i, 0 otherwise. Note that $\tilde{\tau_i}\big(f_{\tau_i}(\alpha)\big)=\alpha$, so that $H=f_{\tau_i}^{-1}\big(\tilde{\tau_i}^{-1}(H)\big)$. This implies that $\tilde{\tau_i}^{-1}(H)$ is $\mathbf{\Gamma}$-strategically complete.

(b) As in (a), we consider the continuous map f_τ, so that $\tilde{\tau}\big(f_\tau(\beta)\big)=\beta$ for each $\beta\in 2^\omega$. Here again we get that $f_\tau^{-1}(H)\in\mathbf{\Gamma}(2^\omega)$. Let $\beta\in L$, which gives $\alpha\in H$ with $\beta=\alpha\circ\tau$. As $f_\tau(\beta)\circ\tau=\tilde{\tau}\big(f_\tau(\beta)\big)=\beta$, we get $f_\tau(\beta)\circ\tau=\alpha\circ\tau$ and $f_\tau(\beta)\in H$ by the assumption on H. Conversely, if $f_\tau(\beta)\in H$, then $\beta=\tilde{\tau}\big(f_\tau(\beta)\big)=f_\tau(\beta)\circ\tau\in L$. Thus $f_\tau^{-1}(H)=L$, and $L\in\mathbf{\Gamma}(2^\omega)$.

If $\alpha\in H$, then $\tilde{\tau}(\alpha)=\alpha\circ\tau\in L$. Conversely, assume that $\tilde{\tau}(\alpha)\in L$. Then there is $\beta\in H$ with $\beta\circ\tau=\alpha\circ\tau$. The assumption on H implies that $\alpha\in H$. Thus $H=\tilde{\tau}^{-1}(L)$ and L is $\mathbf{\Gamma}$-strategically complete. $\square$

Lemma 5.2.2 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, and $A\subseteq 2^\omega$. Then $A\in\mathbf{\Gamma}(2^\omega)$ if and only if there is $B\in\mathbf{\Gamma}(\omega^\omega)$ with $A=B\cap 2^\omega$.*

Proof. $\Rightarrow$ Let $r:\omega^\omega\rightarrow 2^\omega$ be a continuous retraction. We just have to set $B:=r^{-1}(A)$.

$\Leftarrow$ Let $i:2^\omega\rightarrow\omega^\omega$ be the canonical injection. Then $A=i^{-1}(B)\in\mathbf{\Gamma}(2^\omega)$. $\square$

This lemma shows that the notation $\mathbf{\Gamma}_u$ in Theorem 5.1.3 will not create any trouble, since it is equivalent to the one in Definition 5.1.2.

Notation. The following notation can essentially be found in [Lo-SR2] (after Lemma 2.5). Let $\mathcal{R}$ be the least set of functions from 2^ω into itself which contains the functions ρ_0^η, the functions $\tilde{\tau_i}$ for $i\geq 1$, and is closed under composition. By Lemma 5.1.6 and Theorem 5.1.7, each $\rho\in\mathcal{R}$ is an independent η-function for some η called the $order$ $o(\rho)$ of ρ.

Definition 5.2.3 *Let $u\in\mathcal{D}$. A set $H\subseteq 2^\omega$ is $strongly$ u-$strategically$ $complete$ if for each $\rho\in\mathcal{R}$ of order η, $\rho^{-1}(H)$ is $\mathbf{\Gamma}_{u^\eta}$-strategically complete and ccs.*

Theorem 5.2.4 *Let $u\in\mathcal{D}$. Then there exists a strongly u-strategically complete set $H_u\subseteq 2^\omega$. In particular, H_u is $\mathbf{\Gamma}_u$-complete and ccs.*

Proof. We will check that the sets H_u given by Theorem 2.7 in [Lo-SR2] essentially work, even if we change them.

The construction is by induction on $u\in\mathcal{D}$. Let us say that u is $nice$ if it satisfies the conclusion of the theorem. By Proposition 5.1.5, it is enough to prove that 0^∞ is nice, that $u(0)1u$ is nice if u is nice, that u^η is nice if u is nice and $\eta<\omega_1$, and that $12<u_p>$ is nice if the u_p's are nice.

• We set $H_{0^\infty}:=\emptyset$, which is clearly strongly 0^∞-strategically complete.

• Assume that u is nice. We set $H_{u(0)1u}:=\neg H_u$, which is strongly $u(0)1u$-strategically complete. Indeed, if $u(0)=0$, then $\mathbf{\Gamma}_{(u(0)1u)^\eta}=\mathbf{\Gamma}_{u(0)1u}=\check{\mathbf{\Gamma}}_u=\check{\mathbf{\Gamma}}_{u^\eta}$. If $u(0)\geq 1$, then

$$\mathbf{\Gamma}_{(u(0)1u)^\eta}=\mathbf{\Gamma}_{(1+\eta+(u(0)-1))1u^\eta}=\check{\mathbf{\Gamma}}_{u^\eta}$$

since $u^\eta(0)=1+\eta+\big(u(0)-1\big)$.

• Assume that u is nice, and let $\eta<\omega_1$. We set $H_{u^\eta}:=(\rho_0^\eta)^{-1}(H_u)$, which is strongly u^η-strategically complete. Indeed, let $\rho\in\mathcal{R}$ of order ξ. Then $\rho^{-1}(H_{u^\eta})=(\rho_0^\eta\circ\rho)^{-1}(H_u)$ is $\mathbf{\Gamma}_{u^{\xi+\eta}}$-strategically complete and compatible with comeager sets since u is nice and $\rho_0^\eta\circ\rho$ is in $\mathcal{R}$ of order $\xi+\eta$. It remains to notice that $(u^\eta)^\xi=u^{\xi+\eta}$, which is clear by induction on u and by definition of the ordinal subtraction.

• Assume that the u_p's are nice. We set

$$\alpha\in H_{12<u_p>}\ \Leftrightarrow\ \begin{cases}\alpha_0=0^\infty\wedge\alpha_1\in H_{u_0}\\ \text{or}\\ \exists m\in\omega\ \ \alpha_0(m)=1\wedge\forall l<m\ \ \alpha_0(l)=0\wedge\alpha_{(m)_0+2}\in H_{u_{((m)_0+2)_0+1}}.\end{cases}$$

- Recall that $\mathbf{\Gamma}_{12<u_p>}=S_1(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{u_p},\mathbf{\Gamma}_{u_0})$. We set $H'_0:=\{\alpha\in 2^\omega\mid\alpha_1\in H_{u_0}\}=\tilde{\tau_1}^{-1}(H_{u_0})$, and for $n\geq 2$,

$$H'_n:=\{\alpha\in 2^\omega\mid\alpha_n\in H_{u_{(n)_0+1}}\}=\tilde{\tau_n}^{-1}(H_{u_{(n)_0+1}}),$$

$$C_n:=\{\alpha\in 2^\omega\mid\exists m\in\omega\ \ \alpha_0(m)=1\ \text{ and }\ \forall l<m\ \ \alpha_0(l)=0\ \text{ and }\ (m)_0+2=n\}.$$

Note that $(C_n)_{n\geq 2}$ is a sequence of pairwise disjoint open sets, and $H'_0\in\mathbf{\Gamma}_{u_0}$, $H'_n\in\mathbf{\Gamma}_{u_{(n)_0+1}}$ if $n\geq 2$ by Lemma 5.2.1.(a). Moreover, $H_{12<u_p>}=\bigcup_{n\geq 2}(H'_n\cap C_n)\cup(H'_0\setminus\bigcup_{n\geq 2}C_n)\in\mathbf{\Gamma}_{12<u_p>}(2^\omega)$, by Lemma 5.2.2 and the reduction property for the class of open sets (see 22.16 in [K]).

- Let $\rho\in\mathcal{R}$ of order η. Then $\rho^{-1}(H_{12<u_p>})\in\mathbf{\Gamma}_{(12<u_p>)^\eta}(2^\omega)$, by Proposition 5.1.5.(a) and a retraction argument in the style of the proof of Lemma 5.2.2. Let π be associated with ρ, $\theta_0:\omega\to\omega$ be a one-to-one enumeration of $\pi^{-1}\big(\text{Ran}(\tau_1)\big)$, and, for $n\geq 2$, $\theta_n:\omega\to\omega$ be a one-to-one enumeration of $\pi^{-1}\big(\text{Ran}(\tau_n)\big)$ and $\theta_0^n:\omega\to\omega$ be a one-to-one enumeration of

$$\pi^{-1}\Big(\big\{j\in\text{Ran}(\tau_0)\mid\big(\tau_0^{-1}(j)\big)_0+2=n\big\}\Big).$$

As τ_i is one-to-one, $\text{Ran}(\tau_i)$ is infinite, and $\pi^{-1}\big(\text{Ran}(\tau_i)\big)$ is also infinite since π is onto. This proves the existence of the θ_n's and of the θ_0^n's. Note that the $\text{Ran}(\tau_i)$'s are pairwise disjoint since $0=<0,0>$. This implies that the elements of $\{\text{Ran}(\theta_n)\mid n\neq 1\}\cup\{\text{Ran}(\theta_0^n)\mid n\geq 2\}$ are pairwise disjoint.

- Note that the fact that $\alpha\in H_n^\eta:=\rho^{-1}(H'_n)$ depends only on $\alpha\circ\theta_n$ if $n\neq 1$. We set, for $n\neq 1$,

$$L_n^\eta:=\{\alpha\circ\theta_n\mid\alpha\in H_n^\eta\}.$$

Note that $\rho^{-1}(H'_0)=\rho^{-1}\big(\tilde{\tau_1}^{-1}(H_{u_0})\big)=(\tilde{\tau_1}\circ\rho)^{-1}(H_{u_0})$ is $\mathbf{\Gamma}_{u_0^\eta}$-strategically complete since u_0 is nice and $\tilde{\tau_1}\circ\rho$ is in $\mathcal{R}$ of order η. Similarly, $\rho^{-1}(H'_n)$ is $\mathbf{\Gamma}_{u^\eta_{(n)_0+1}}$-strategically complete if $n\geq 2$. By Lemma 5.2.1.(b), we get that L_0^η is $\mathbf{\Gamma}_{u_0^\eta}$-strategically complete, and L_n^η is $\mathbf{\Gamma}_{u^\eta_{(n)_0+1}}$-strategically complete if $n\geq 2$.

- We set, for $n\geq 2$, $C^\eta_n:=\{\alpha\circ\theta^n_0\mid\exists m\in\omega\ \ \rho(\alpha)_0(m)=1\ \text{ and }\ (m)_0+2=n\}$. Let us prove that C^η_n is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete.

Note first that $\{\alpha\in 2^\omega\mid f(\alpha)\neq 0^\infty\}$ is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete if f is an independent η-function. Indeed, with the notation of Definition 3.3, we can write

$$\{\alpha\in 2^\omega\mid f(\alpha)=0^\infty\}=\bigcap_{m\in\omega}\neg\,\mathcal{C}_m.$$

Moreover, the fact that $\alpha\in\mathcal{C}_m$ depends only of the values of α on $\pi_f^{-1}(\{m\})$.

Assume first that $\eta\geq 1$. As f is an independent η-function, $\mathcal{C}_m$ is $\mathbf{\Pi}^0_{1+\theta_m}$-strategically complete, for some $\theta_m<\eta$ satisfying $\theta_m+1=\eta$ if η is a successor ordinal, $\sup_{m\in\omega}\ \theta_m=\eta$ if η is a limit ordinal. Note that $\eta=\sup_{m\in\omega}\ (\theta_m+1)$. By Lemma 3.7 in [Lo-SR1], $\{\alpha\in 2^\omega\mid f(\alpha)=0^\infty\}$ is $\mathbf{\Pi}^0_{1+\eta}$-strategically complete.

Assume now that $\eta=0$. As in the proof of Theorem 5.1.7 we see that $\{\alpha\in 2^\omega\mid f(\alpha)=0^\infty\}$ is $\mathbf{\Pi}^0_{1+\eta}$-strategically complete.

Now we come back to the C^η_n's. We define $\tau:\omega\to\omega$ by $\tau(k):=<n-2,k>$, so that τ is one-to-one and $\text{Ran}(\tau)=\{m\in\omega\mid(m)_0=n-2\}$. As ρ is an independent η-function, $\tilde{\tau}_0\circ\rho$ and $\tilde{\tau}\circ\tilde{\tau}_0\circ\rho$ are also independent η-functions by Lemma 5.1.6. The previous point shows that

$$L_n:=\{\alpha\in 2^\omega\mid(\tilde{\tau}\circ\tilde{\tau}_0\circ\rho)(\alpha)\neq 0^\infty\}$$

is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete. But

$$\begin{aligned}L_n&=\{\alpha\in 2^\omega\mid\exists k\in\omega\ \ \tilde{\tau}\big((\tilde{\tau}_0\circ\rho)(\alpha)\big)(k)=1\}=\{\alpha\in 2^\omega\mid\exists k\in\omega\ \ (\tilde{\tau}_0\circ\rho)(\alpha)\big(\tau(k)\big)=1\}\\&=\{\alpha\in 2^\omega\mid\exists m\in\omega\ \ (\tilde{\tau}_0\circ\rho)(\alpha)(m)=1\ \text{ and }\ (m)_0+2=n\}\end{aligned}$$

and the fact that $\alpha\in L_n$ depends only on $\alpha\circ\theta^n_0$. By Lemma 5.2.1.(b), we get that C^ξ_n is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete.

- Let $H^*\in\mathbf{\Gamma}_{(12<u_p>)^\eta}(\omega^\omega)$, say $H^*=\bigcup_{n\geq 2}\ (H^*_n\cap C^*_n)\cup(H^*_0\setminus\bigcup_{n\geq 2}\ C^*_n)$, with pairwise disjoint $C^*_n\in\mathbf{\Sigma}^0_{1+\eta}$, $H^*_0\in\mathbf{\Gamma}_{u^\eta_0}$, and without loss of generality $H^*_n\in\mathbf{\Gamma}_{u^\eta_{(n)_0+1}}$. Then Player 2 has for each $n\neq 1$ a winning strategy σ_n in $G(H^*_n,L^\eta_n)$, and for each $n\geq 2$ a winning strategy σ^*_n in $G(C^*_n,C^\eta_n)$. Let then Player 2 plays in $G\big(H^*,\rho^{-1}(H_{u_{12<u_p>}})\big)$ against β by playing his strategies σ_n, σ^*_n at the right places (the ranges of θ_n and θ^n_0 respectively) against this same β, independently, and by playing 0 out of these ranges. The result is some α such that $\alpha\circ\theta_n$ wins against β in $G(H^*_n,L^\eta_n)$ and $\alpha\circ\theta^n_0$ wins against β in $G(C^*_n,C^\eta_n)$. This wins, for $\alpha\in\rho^{-1}(H'_n)$ just in case $\beta\in H^*_n$, and $\rho(\alpha)_0$ takes value 1 on some m with $(m)_0+2=n$ just in case $\beta\in C^*_n$. But as the C^*_n are pairwise disjoint, there is at most one n in $\{(m)_0+2\mid\rho(\alpha)_0(m)=1\}$, and $\alpha\in\rho^{-1}(C_n)$ just in case $\beta\in C^*_n$. Thus $\rho^{-1}(H_{12<u_p>})$ is $\mathbf{\Gamma}_{(12<u_p>)^\eta}$-strategically complete.

- It remains to see that $\rho^{-1}(H_{12<u_p>})$ is ccs. So let $\alpha_0\in d^\omega$ and $F:2^\omega\rightarrow (d^\omega)^{d-1}$ satisfying the conclusion of Lemma 2.4.(b).

$\circ$ Let $N\geq 1$ and $M\in\omega$. Then $\rho(\alpha)_N\in H_{u_M}\Leftrightarrow (\tilde{\tau_N}\circ\rho)(\alpha)\in H_{u_M}\Leftrightarrow \alpha\in(\tilde{\tau_N}\circ\rho)^{-1}(H_{u_M})$. As $N\geq 1$, $\tilde{\tau_N}\circ\rho$ is in $\mathcal{R}$, and $(\tilde{\tau_N}\circ\rho)^{-1}(H_{u_M})$ is ccs since u_M is nice. Thus $\rho(\alpha)_N\in H_{u_M}$ if and only if $\rho\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)_N\in H_{u_M}$.

$\circ$ Recall the notation before Lemma 2.4. We define $q:\omega^{<\omega}\setminus\{\emptyset\}\rightarrow\omega$ as follows:

$$q(t):=\begin{cases} t(0) \text{ if } |t|=1,\\ \\ <t(|t|-1),q(t^-)> \text{ if } |t|\geq 2.\end{cases}$$

$\circ$ Let us prove that $\tilde{\tau}_s(\alpha)(n)=\alpha(<q\big((n)_0 s\big),(n)_1>)$ for each $s\in(\omega\setminus\{0\})^{<\omega}$.

We argue by induction on $|s|$. So assume that the result is proved for $|s|\leq l$, which is the case for $l=0$. Assume that $|s|=l+1$. We get

$$\tilde{\tau}_s(\alpha)(n)=\tilde{\tau}_{s|l}\big(\tilde{\tau_{s(l)}}(\alpha)\big)(n)=\tilde{\tau_{s(l)}}(\alpha)(<q\big((n)_0(s|l)\big),(n)_1>)=\alpha\big(\tau_{s(l)}(<q\big((n)_0(s|l)\big),(n)_1>)\big)$$

$$=\alpha\big(\big\langle <s(l),q\big((n)_0(s|l)\big)>,(n)_1\big\rangle\big)=\alpha(<q\big((n)_0 s\big),(n)_1>).$$

$\circ$ Let us prove that $(\rho_0\circ\tilde{\tau}_s)(\alpha)=(\rho_0\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$ for each $s\in(\omega\setminus\{0\})^{<\omega}$ and each $\alpha\in 2^\omega$. This comes from the following equivalences:

$$\begin{aligned}(\rho_0\circ\tilde{\tau}_s)(\alpha)(n)=0 &\Leftrightarrow \exists m\in\omega\ \ \tilde{\tau}_s(\alpha)(<n,m>)=1\Leftrightarrow\exists m\in\omega\ \ \alpha(<q(ns),m>)=1\\ &\Leftrightarrow \exists m'\in\omega\ \ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)(<q(ns),m'>)=1\\ &\Leftrightarrow (\rho_0\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(n)=0.\end{aligned}$$

$\circ$ Let us prove that $(\rho_0^\eta\circ\tilde{\tau}_s)(\alpha)=(\rho_0^\eta\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$ for each $1\leq\eta<\omega_1$, each $s\in(\omega\setminus\{0\})^{<\omega}$ and each $\alpha\in 2^\omega$.

We argue by induction on η. For $\eta=1$, this comes from the previous point. If $\theta\geq 1$ and $\eta=\theta+1$, then this comes from the fact that $\rho_0^\eta=\rho_0\circ\rho_0^\theta$. If η is a limit ordinal and m is an integer, then

$$\begin{aligned}&(\rho_0^\eta\circ\tilde{\tau}_s)(\alpha)(m)\\ &=\rho_0^\eta\big(\tilde{\tau}_s(\alpha)\big)(m)=\rho_0^{(0,m+1)}\big(\tilde{\tau}_s(\alpha)\big)(m)\\ &=(\rho_0^{(m,m+1)}\circ ...\circ\rho_0^{(1,2)})\Big(\rho_0^{(0,1)}\big(\tilde{\tau}_s(\alpha)\big)\Big)(m)=(\rho_0^{(m,m+1)}\circ ...\circ\rho_0^{(1,2)})\Big(\rho_0^{\theta_0^\eta}\big(\tilde{\tau}_s(\alpha)\big)\Big)(m)\\ &=(\rho_0^{(m,m+1)}\circ ...\circ\rho_0^{(1,2)})\Bigg(\rho_0^{\theta_0^\eta}\bigg(\tilde{\tau}_s\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)\bigg)\Bigg)(m)=(\rho_0^\eta\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m).\end{aligned}$$

$\circ$ Note that $\rho(\alpha)_0 = 0^\infty \Leftrightarrow \alpha \in (\tilde{\tau}_0 \circ \rho)^{-1}(\{0^\infty\})$. Let us prove that $(\tilde{\tau}_0 \circ \rho)^{-1}(\{0^\infty\})$ is ccs.

We can write $\rho = \circ_{j\leq l}\ \rho^j$, where l is an integer and each ρ^j is either of the form ρ_0^η, or one of the $\tilde{\tau}_i$'s for $i\geq 1$. By the previous point, we may assume that that each ρ^j is either $\rho_0^0 = \mathrm{Id}_{2^\omega}$, or one of the $\tilde{\tau}_i$'s for $i\geq 1$. So there is $s\in(\omega\setminus\{0\})^{<\omega}$ such that $\rho = \tilde{\tau}_s$. We get

$$\begin{aligned}
\alpha \notin (\tilde{\tau}_0 \circ \rho)^{-1}(\{0^\infty\}) &\Leftrightarrow \exists m\in\omega\ \ (\tilde{\tau}_0 \circ \rho)(\alpha)(m) = 1 \Leftrightarrow \exists m\in\omega\ \ \rho(\alpha)\big(\tau_0(m)\big) = 1\\
&\Leftrightarrow \exists m\in\omega\ \ \tilde{\tau}_s(\alpha)(< 0, m >) = 1 \Leftrightarrow \exists m\in\omega\ \ \alpha(< q(0s), m >) = 1\\
&\Leftrightarrow \exists m\in\omega\ \ \alpha\big(p(q(0s), m)\big) = 1\\
&\Leftrightarrow \exists m'\in\omega\ \ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\big(p(q(0s), m')\big) = 1\\
&\Leftrightarrow \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big) \notin (\tilde{\tau}_0 \circ \rho)^{-1}(\{0^\infty\}).
\end{aligned}$$

Thus $\rho(\alpha)_0 = 0^\infty \Leftrightarrow \rho\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)_0 = 0^\infty$.

$\circ$ It remains to see that if $\rho(\alpha)_0 \neq 0^\infty$ and m_α is minimal with $\rho(\alpha)_0(m_\alpha) = 1$, then

$$(m_\alpha)_0 = (m_{\mathcal{S}(\alpha_0\Delta F_0(\alpha))})_0.$$

As in the previous point we may assume that there is $s\in(\omega\setminus\{0\})^{<\omega}$ such that $\rho = \tilde{\tau}_s$. The computations of the previous point show that $\rho(\alpha)_0(m) = \alpha(< q(0s), m >)$ for each integer m. Note that

$$n_\alpha := < q(0s), m_\alpha > = \min\{n\in\omega \mid \alpha(n) = 1 \wedge (n)_0 = q(0s)\}$$

since $< q(0s), . >$ is increasing, and similarly

$$< q(0s), m_{\mathcal{S}(\alpha_0\Delta F_0(\alpha))} > = \min\{m\in\omega \mid \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)(m) = 1 \wedge (m)_0 = q(0s)\}.$$

But

$$B_\alpha[\{n\in\omega \mid \alpha(n) = 1 \ \text{ and } \ (n)_0 = q(0s)\}] = \{m\in\omega \mid \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)(m) = 1 \ \text{ and } \ (m)_0 = q(0s)\}$$

since B_α is a bijection satisfying $(n)_0 = \big(B_\alpha(n)\big)_0$. As B_α is increasing we get

$$B_\alpha(n_\alpha) = < q(0s), m_{\mathcal{S}(\alpha_0\Delta F_0(\alpha))} > .$$

Thus $(m_{\mathcal{S}(\alpha_0\Delta F_0(\alpha))})_0 = \Big(\big(B_\alpha(n_\alpha)\big)_1\Big)_0 = \big((n_\alpha)_1\big)_0 = (m_\alpha)_0$ and we are done. $\square$

Corollary 5.2.5 *Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets. Then there is $C_{\mathbf{\Gamma}} \subseteq 2^\omega$ which is $\mathbf{\Gamma}$-complete and ccs.*

Proof. By Theorem 5.1.3 there is $u \in \mathcal{D}$ such that $\mathbf{\Gamma}(\omega^\omega) = \mathbf{\Gamma}_u(\omega^\omega)$. By Theorem 5.2.4 there is $H_u \subseteq 2^\omega$ which is strongly $\mathbf{\Gamma}_u$-strategically complete. It is clear that $C_{\mathbf{\Gamma}} := H_u$ is suitable. $\square$

Now we can prove Theorem 1.7.(1). But we need some more material to prove Theorem 1.7.(2).

Definition 5.2.6 *(a) A set $U\subseteq 2^\omega$ is $strongly\ ccs$ if for each $s\in(\omega\setminus\{0\})^{<\omega}$ the set $\tilde{\tau}_s^{-1}(U)$ is ccs.*

(b) Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, and $U_0,U_1\in\mathbf{\Gamma}(2^\omega)$ disjoint. We say that (U_0,U_1) $is\ complete\ for\ pairs\ of\ disjoint$ $\mathbf{\Gamma}$ $sets$ if for any pair (A_0,A_1) of disjoint $\mathbf{\Gamma}$ subsets of ω^ω there is $f:\omega^\omega\rightarrow 2^\omega$ continuous such that $A_\varepsilon=f^{-1}(U_\varepsilon)$ for each $\varepsilon\in 2$. Similarly, we can define the notion of a sequence $(U_p)_{p\geq 1}$ complete for sequences of pairwise disjoint $\mathbf{\Gamma}$ sets.

Lemma 5.2.7 *(a) There is (U_1,U_2) complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets with U_ε strongly ccs, and such that for each $s\in(\omega\setminus\{0\})^{<\omega}$ there is a pair (O_1,O_2) of ccs $\mathbf{\Sigma}^0_1$ sets reducing the pair $\big(\tilde{\tau}_{1s1}^{-1}(U_1\cup U_2),\tilde{\tau}_{1s2}^{-1}(U_1\cup U_2)\big)$.*
(b) There is $(U_p)_{p\geq 1}$ complete for sequences of pairwise disjoint $\mathbf{\Sigma}^0_1$ sets with U_p strongly ccs, and such that for each $s\in(\omega\setminus\{0\})^{<\omega}$ there is a sequence $(O^\varepsilon_p)_{\varepsilon\in\{1,2\},p\geq 1}$ of ccs $\mathbf{\Sigma}^0_1$ sets reducing $\big(\tilde{\tau}_{s\varepsilon}^{-1}(U_p)\big)_{\varepsilon\in\{1,2\},p\geq 1}$.

Proof. (a) Recall the definition of H_1 after Definition 3.3: $H_1:=\{0^\infty\}$. We saw that $H_1\in\mathbf{\Pi}^0_1(2^\omega)$ and is $\mathbf{\Pi}^0_1$-complete. We set $U:=\neg H_1$, so that U is $\mathbf{\Sigma}^0_1$-complete. Let (A_1,A_2) be a pair of disjoint $\mathbf{\Sigma}^0_1$ subsets of ω^ω. As U is complete there are $f_1,f_2:\omega^\omega\rightarrow 2^\omega$ continuous such that $A_\varepsilon=f_\varepsilon^{-1}(U)$ for each $\varepsilon\in\{1,2\}$. We define $f:\omega^\omega\rightarrow 2^\omega$ by

$$f(\alpha)\big(\langle <\varepsilon,(k)_0>,(k)_1\rangle\big):=\begin{cases} f_\varepsilon(\alpha)(k) \text{ if } \varepsilon\in\{1,2\},\\ 0 \text{ otherwise,}\end{cases}$$

so that f is continuous and $f_\varepsilon=\tilde{\tau}_\varepsilon\circ f$. Now $A_\varepsilon=f^{-1}\big(\tilde{\tau}_\varepsilon^{-1}(U)\big)$ and $\big(\tilde{\tau}_1^{-1}(U),\tilde{\tau}_2^{-1}(U)\big)$ is complete for pairs of $\mathbf{\Sigma}^0_1$ sets (not necessarily disjoint). Note that

$$\begin{aligned}\tilde{\tau}_\varepsilon^{-1}(U)&=\big\{\alpha\in 2^\omega\mid \exists k\in\omega\ \ \alpha\big(\langle <\varepsilon,(k)_0>,(k)_1\rangle\big)=1\big\}\\ &=\big\{\alpha\in 2^\omega\mid \exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon\wedge\alpha(N)=1\big\}.\end{aligned}$$

We set $V_\varepsilon:=\Big\{\alpha\in 2^\omega\mid \exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon\wedge\alpha(N)=1\wedge\forall l<N\ \ \Big(\big((l)_0\big)_0\notin\{1,2\}\vee\alpha(l)=0\Big)\Big\}$.
Note that $V_i\in\mathbf{\Sigma}^0_1$ and (V_1,V_2) reduces $\big(\tilde{\tau}_1^{-1}(U),\tilde{\tau}_2^{-1}(U)\big)$. Thus

$$\alpha\in A_\varepsilon\Leftrightarrow f(\alpha)\in\tilde{\tau}_\varepsilon^{-1}(U)\Leftrightarrow f(\alpha)\in\tilde{\tau}_\varepsilon^{-1}(U)\setminus\tilde{\tau}_{3-\varepsilon}^{-1}(U)\Leftrightarrow f(\alpha)\in V_\varepsilon$$

and (V_1,V_2) is complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets. Recall the definition of τ_0 before Lemma 5.2.1. We set $U_\varepsilon:=\tilde{\tau}_0^{-1}(V_\varepsilon)$, which defines a pair of disjoint $\mathbf{\Sigma}^0_1$ sets. Now $g(\alpha):=<\alpha,\alpha,...>$ defines $g:2^\omega\rightarrow 2^\omega$ continuous. Note that $\alpha\in A_\varepsilon\Leftrightarrow f(\alpha)\in V_\varepsilon\Leftrightarrow\tilde{\tau}_0\Big(g\big(f(\alpha)\big)\Big)\in V_\varepsilon\Leftrightarrow g\big(f(\alpha)\big)\in U_\varepsilon$, which shows that (U_1,U_2) is complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets.

Fix $s\in(\omega\setminus\{0\})^{<\omega}$. The proof of Theorem 5.2.4 shows that $\tilde{\tau}_s(\alpha)(n)=\alpha\big(<q\big((n)_0s\big),(n)_1>\big)$. We get

$$\begin{aligned}\tilde{\tau}_s^{-1}(U_\varepsilon)=\Big\{\alpha\in 2^\omega\mid \exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon\wedge\alpha(<q(0s),N>)=1\wedge\\ \forall l<N\ \ \Big(\big((l)_0\big)_0\notin\{1,2\}\vee\alpha(<q(0s),l>)=0\Big)\Big\}.\end{aligned}$$

Thus

$$\tilde{\tau}_s^{-1}(U_\varepsilon)=\Big\{\alpha\in 2^\omega \mid \exists M\in\omega\ \ \big(((M)_1)_0\big)_0=\varepsilon \wedge (M)_0=q(0s)\wedge \alpha(M)=1\ \wedge$$
$$\forall l<M\ \ \Big(\big(((M)_1)_0\big)_0\notin\{1,2\}\vee (l)_0\neq q(0s)\vee \alpha(l)=0\Big)\Big\}.$$

Recall the conclusion of Lemma 2.4.(b). The bijection B_α induces an increasing bijection between $\Big\{M\in\omega \mid \big(((M)_1)_0\big)_0\in\{1,2\}\wedge (M)_0=q(0s)\wedge \alpha(M)=1\Big\}$ and

$$\Big\{M'\in\omega \mid \big(((M')_1)_0\big)_0\in\{1,2\}\wedge (M')_0=q(0s)\wedge \mathcal{S}\big(\alpha_0\Delta F(\alpha)\big)(M')=1\Big\}$$

since $(M)_0=\big(B_\alpha(M)\big)_0$ and $\big((M)_1\big)_0=\Big(\big(B_\alpha(M)\big)_1\Big)_0$. A second application of this shows that $\tilde{\tau}_s^{-1}(U_\varepsilon)$ is ccs. Thus U_ε strongly ccs. Note that

$$\tilde{\tau}_{1s\varepsilon}^{-1}(U_1\cup U_2)=\Big\{\alpha\in 2^\omega \mid \exists M\in\omega\ \big(((M)_1)_0\big)_0\in\{1,2\}\wedge (M)_0=q(01s\varepsilon)\wedge \alpha(M)=1\Big\}.$$

We set

$$O_\varepsilon:=\Big\{\alpha\in 2^\omega \mid \exists M\in\omega\ \big(((M)_1)_0\big)_0\in\{1,2\}\wedge (M)_0=q(01s\varepsilon)\wedge \alpha(M)=1\ \wedge$$
$$\forall l<M\ \ \Big(\big(((l)_1)_0\big)_0\notin\{1,2\}\vee (l)_0\notin\{q(01s1),q(01s2)\}\vee \alpha(l)=0\Big)\Big\}$$

This defines a pair of $\mathbf{\Sigma}^0_1$ sets reducing $\big(\tilde{\tau}_{1s1}^{-1}(U_1\cup U_2),\tilde{\tau}_{1s2}^{-1}(U_1\cup U_2)\big)$. We check that they are ccs as for $\tilde{\tau}_s^{-1}(U_\varepsilon)$.

(b) The proof is completely similar to that of (a). $\square$

The following result is a consequence of Theorem 1.9 and Lemmas 1.11, 1.23 in [Lo1], and of Theorem 3 in [Lo-SR3]:

Theorem 5.2.8 *Let $\mathbf{\Gamma}$ be a self-dual Wadge class of Borel sets. Then there is a non self-dual Wadge class of Borel sets $\mathbf{\Gamma}'$ such that $\mathbf{\Gamma}(\omega^\omega)=\Delta(\mathbf{\Gamma}')(\omega^\omega)$, $\mathbf{\Gamma}'$ does not have the separation property, and one of the following holds:*
(1) There is $\overline{u}\in\mathcal{D}$ such that

$$\mathbf{\Gamma}'(\omega^\omega)=\Big\{(A_0\cap C_0)\cup(A_1\cap C_1)\mid A_0,\neg A_1\in\mathbf{\Gamma}_{\overline{u}}(\omega^\omega)\wedge C_0,C_1\in\mathbf{\Sigma}^0_1(\omega^\omega)\wedge C_0\cap C_1=\emptyset\Big\}.$$

(2) There is $\big((u')_p\big)_{p\geq 1}\in\mathcal{D}^\omega$ such that $\big(\mathbf{\Gamma}_{(u')_p}(\omega^\omega)\big)_{p\geq 1}$ is strictly increasing and

$$\mathbf{\Gamma}'(\omega^\omega)=\Big\{\bigcup_{p\geq 1}\ (A_p\cap C_p)\mid A_p\in\mathbf{\Gamma}_{(u')_p}(\omega^\omega)\wedge C_p\in\mathbf{\Sigma}^0_1(\omega^\omega)\wedge C_p\cap C_q=\emptyset \textit{ if } p\neq q\Big\}.$$

Lemma 5.2.9 *Let $\mathbf{\Gamma}'$ be as in the statement of Theorem 5.2.8. Then there are $C^0,C^1\in\mathbf{\Gamma}'(2^\omega)$ disjoint, ccs, and not separable by a $\Delta(\mathbf{\Gamma}')$ set.*

Proof. (1) Lemma 5.2.7.(a) gives (U_1,U_2) complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets with U_ε strongly ccs, and such that for each $s\in(\omega\setminus\{0\})^{<\omega}$ there is a pair (O_1,O_2) of ccs $\mathbf{\Sigma}^0_1$ sets reducing the pair $\big(\tilde\tau^{-1}_{1s1}(U_1\cup U_2),\tilde\tau^{-1}_{1s2}(U_1\cup U_2)\big)$. Theorem 5.2.4 gives $H_{\overline{u}}\subseteq 2^\omega$ which is $\mathbf{\Gamma}_{\overline{u}}$-complete and strongly ccs. We set $H:=\big(\tilde\tau^{-1}_2(H_{\overline{u}})\cap\tilde\tau^{-1}_1(U_1)\big)\cup\big(\tilde\tau^{-1}_3(\neg H_{\overline{u}})\setminus\tilde\tau^{-1}_1(U_2)\big)$ and, for $\varepsilon\in\{1,2\}$, $E_\varepsilon:=\tilde\tau^{-1}_\varepsilon(H)$. Finally, we set $C^\varepsilon:=(O_\varepsilon\cap E_\varepsilon)\cup(O_{3-\varepsilon}\setminus E_{3-\varepsilon})$.

• We set, for $\varepsilon,j\in\{1,2\}$, $A^\varepsilon_1:=\tilde\tau^{-1}_{2\varepsilon}(H_{\overline{u}})$, $A^\varepsilon_2:=\tilde\tau^{-1}_{3\varepsilon}(\neg H_{\overline{u}})$, $F^\varepsilon_j:=\tilde\tau^{-1}_{1\varepsilon}(U_j)$, so that

$$E_\varepsilon=(A^\varepsilon_1\cap F^\varepsilon_1)\cup(A^\varepsilon_2\cap F^\varepsilon_2).$$

Note that

$$\begin{aligned}C^\varepsilon&=(A^\varepsilon_1\cap F^\varepsilon_1\cap O_\varepsilon)\cup(A^\varepsilon_2\cap F^\varepsilon_2\cap O_\varepsilon)\cup(\neg A^{3-\varepsilon}_1\cap F^{3-\varepsilon}_1\cap O_{3-\varepsilon})\cup(\neg A^{3-\varepsilon}_2\cap F^{3-\varepsilon}_2\cap O_{3-\varepsilon})\\ &=\Big(\big((A^\varepsilon_1\cap F^\varepsilon_1\cap O_\varepsilon)\cup(\neg A^{3-\varepsilon}_2\cap F^{3-\varepsilon}_2\cap O_{3-\varepsilon})\big)\cap\big((F^\varepsilon_1\cap O_\varepsilon)\cup(F^{3-\varepsilon}_2\cap O_{3-\varepsilon})\big)\Big)\cup\\ &\quad\Big(\big((A^\varepsilon_2\cap F^\varepsilon_2\cap O_\varepsilon)\cup(\neg A^{3-\varepsilon}_1\cap F^{3-\varepsilon}_1\cap O_{3-\varepsilon})\big)\cap\big((F^\varepsilon_2\cap O_\varepsilon)\cup(F^{3-\varepsilon}_1\cap O_{3-\varepsilon})\big)\Big),\end{aligned}$$

and that $F^\varepsilon_1\cap O_\varepsilon$, $F^{3-\varepsilon}_2\cap O_{3-\varepsilon}$, $F^\varepsilon_2\cap O_\varepsilon$, $F^{3-\varepsilon}_1\cap O_{3-\varepsilon}$ are pairwise disjoint open subsets of 2^ω. By Lemma 5.2.2 and the reduction property for $\mathbf{\Sigma}^0_1$ we can write C^ε as the intersection of 2^ω with

$$\Big(\big((\mathcal{A}^\varepsilon_1\cap\mathcal{O}^\varepsilon_1)\cup(\neg\mathcal{A}^{3-\varepsilon}_2\cap\mathcal{O}^{3-\varepsilon}_2)\big)\cap(\mathcal{O}^\varepsilon_1\cup\mathcal{O}^{3-\varepsilon}_2)\Big)\cup\Big(\big((\mathcal{A}^\varepsilon_2\cap\mathcal{O}^\varepsilon_2)\cup(\neg\mathcal{A}^{3-\varepsilon}_1\cap\mathcal{O}^{3-\varepsilon}_1)\big)\cap(\mathcal{O}^\varepsilon_2\cup\mathcal{O}^{3-\varepsilon}_1)\Big),$$

where $\mathcal{A}^\varepsilon_1,\neg\mathcal{A}^\varepsilon_2\in\mathbf{\Gamma}_{\overline{u}}(\omega^\omega)$ and $\mathcal{O}^\varepsilon_j$ are four pairwise disjoint open subsets of ω^ω. By Lemma 1.4.(b) in [Lo1], $(\mathcal{A}^\varepsilon_1\cap\mathcal{O}^\varepsilon_1)\cup(\neg\mathcal{A}^{3-\varepsilon}_2\cap\mathcal{O}^{3-\varepsilon}_2),\neg\big((\mathcal{A}^\varepsilon_2\cap\mathcal{O}^\varepsilon_2)\cup(\neg\mathcal{A}^{3-\varepsilon}_1\cap\mathcal{O}^{3-\varepsilon}_1)\big)\in\mathbf{\Gamma}_{\overline{u}}(\omega^\omega)$, so that $C^\varepsilon\in\mathbf{\Gamma}'(2^\omega)$, by Lemma 5.2.2 again.

• It is clear that C^1 and C^2 are disjoint and ccs.

• Assume, towards a contradiction, that $D\in\Delta(\mathbf{\Gamma}')$ separates C^1 from C^2. Let $D_1,D_2\in\mathbf{\Gamma}'(\omega^\omega)$ disjoint. As H is complete we get $f_\varepsilon:\omega^\omega\to 2^\omega$ continuous such that $D_\varepsilon=f^{-1}_\varepsilon(H)$. We define $f:\omega^\omega\to 2^\omega$ by

$$f(\alpha)\big(\langle<\varepsilon,(k)_0>,(k)_1\rangle\big):=\begin{cases}f_\varepsilon(\alpha)(k)\text{ if }\varepsilon\in\{1,2\},\\ 0\text{ otherwise,}\end{cases}$$

so that $\big(f(\alpha)\big)_\varepsilon=f_\varepsilon(\alpha)$. Then f is continuous and $D_\varepsilon=f^{-1}(E_\varepsilon)$. Note that $E_\varepsilon\setminus E_{3-\varepsilon}\subseteq C^\varepsilon$. This implies that $\alpha\in D_1\Leftrightarrow f(\alpha)\in E_1\Leftrightarrow f(\alpha)\in E_1\setminus E_2\Rightarrow f(\alpha)\in C_1\subseteq D$. Similarly, $D_2\subseteq f^{-1}(\neg D)$, and $f^{-1}(D)\in\Delta(\mathbf{\Gamma}')(\omega^\omega)$ separates D_1 from D_2. Thus $\mathbf{\Gamma}'$ has the separation property, which is absurd.

(2) Lemma 5.2.7.(b) gives $(U_p)_{p\geq 1}$ complete for sequences of pairwise disjoint $\mathbf{\Sigma}^0_1$ sets with U_p strongly ccs, and such that for each $s\in(\omega\setminus\{0\})^{<\omega}$ there is a sequence $(O^\varepsilon_p)_{\varepsilon\in\{1,2\},p\geq 1}$ of ccs $\mathbf{\Sigma}^0_1$ sets reducing $\big(\tilde\tau^{-1}_{s\varepsilon}(U_p)\big)_{\varepsilon\in\{1,2\},p\geq 1}$. Theorem 5.2.4 gives $H_{(u')_p}\subseteq 2^\omega$ which is $\mathbf{\Gamma}_{(u')_p}$-complete and strongly ccs. We set $H:=\bigcup_{p\geq 1}\ \big(\tilde\tau^{-1}_{2p}(H_{(u')_p})\cap\tilde\tau^{-1}_1(U_p)\big)$ and, for $\varepsilon\in\{1,2\}$, $E_\varepsilon:=\tilde\tau^{-1}_\varepsilon(H)$.

We also set $A_p^\varepsilon := \tilde{\tau}_{(2p)\varepsilon}^{-1}(H_{(u')_p})$, $F_p^\varepsilon := \tilde{\tau}_{1\varepsilon}^{-1}(U_p)$, so that $E_\varepsilon = \bigcup_{p\geq 1}\ (A_p^\varepsilon \cap F_p^\varepsilon)$. Finally, we set $C^\varepsilon := (A_1^\varepsilon \cap O_1^\varepsilon) \cup \bigcup_{p\geq 1}\ \big((O_p^{3-\varepsilon} \setminus A_p^{3-\varepsilon}) \cup (A_{p+1}^\varepsilon \cap O_{p+1}^\varepsilon)\big)$.

Note that $C^\varepsilon \in \mathbf{\Gamma}'(2^\omega)$ since $\big(\mathbf{\Gamma}_{(u')_p}(\omega^\omega)\big)_{p\geq 1}$ is strictly increasing, using again Lemma 5.2.2, the generalized reduction property for $\mathbf{\Sigma}_1^0$ (see 22.16 in [K]), and Lemma 1.4.(b) in [Lo1]. Here again, $E_\varepsilon \setminus E_{3-\varepsilon} \subseteq C^\varepsilon$ and we conclude as in (1). $\square$

Proof of Theorem 1.7. It is clear that Proposition 2.2, Lemmas 2.3, 2.6, Corollary 5.2.5, Lemma 5.2.9 and Theorem 3.1 imply Theorem 1.7, if we set $\mathbb{S}_{\mathbf{\Gamma}}^d := S_{C_{\mathbf{\Gamma}}}^d$ and $\mathbb{S}_{\mathbf{\Gamma}}^\varepsilon := S_{C^\varepsilon}^d$. $\square$

6 The proof of Theorem 1.8

We first introduce an operator in the spirit of Φ defined before Theorem 4.2.2, but in dimension one. Another important difference to notice is the following. In Theorem 4.2.2, (f) for example, S is in a boldface class, while A_0 and A_1 are in a lightface class. The same phenomenon will hold in the case of Wadge classes, and in the new operator we introduce we have boldface conditions (for example, we do not ask γ' to be $\Delta_1^1(\beta)$). We code the Borel classes, and define an operator Φ_1 on $\omega^\omega \times \omega^\omega$ to do it. Recall the definition of Seq before Lemma 2.3. We set

$$W_0 := \Big\{(\beta,\gamma) \in \omega^\omega \times W^{\omega^\omega} \mid \Big(\beta(0) \in \text{Seq} \wedge C_\gamma^{\omega^\omega} = \big\{\delta \in \omega^\omega \mid \mathcal{I}^{-1}\big(\beta(0)\big) \subseteq \delta\big\}\Big) \vee \Big(\beta(0) \notin \text{Seq} \wedge C_\gamma^{\omega^\omega} = \emptyset\Big)\Big\},$$

$$\Phi_1(A) := A \cup W_0 \cup \Big\{(\beta,\gamma) \in \omega^\omega \times W^{\omega^\omega} \mid \exists \gamma' \in \omega^\omega\ \ \forall n \in \omega\ \ \big((\beta)_n, (\gamma')_n\big) \in A \text{ and } \neg C_\gamma^{\omega^\omega} = \bigcup_{n\in\omega}\ C_{(\gamma')_n}^{\omega^\omega}\Big\}.$$

In the sequel, we will denote $\Phi_1^{<\xi} := \bigcup_{\eta<\xi}\ \Phi_1^\eta$.

Lemma 6.1 *Let $1 \leq \xi < \omega_1$ and $B \subseteq \omega^\omega$. Then $B \in \mathbf{\Pi}_\xi^0$ if and only if there is $(\beta,\gamma) \in \Phi_1^\xi$ such that $C_\gamma^{\omega^\omega} = B$.*

Proof. Note first that $B = N_s := \{\delta \in \omega^\omega \mid s \subseteq \delta\}$ for some $s \in \omega^{<\omega}$ or $B = \emptyset$ if and only if there is $(\beta,\gamma) \in W_0 = \Phi_1^0$ with $C_\gamma^{\omega^\omega} = B$. Then

$$\begin{aligned} B \in \mathbf{\Pi}_1^0 &\Leftrightarrow \exists (s_n)_{n\in\omega} \in (\omega^{<\omega})^\omega\ \ \neg B = \textstyle\bigcup_{n\in\omega}\ N_{s_n} \vee \neg B = \emptyset \\ &\Leftrightarrow \exists \beta, \gamma' \in \omega^\omega\ \ \forall n \in \omega\ \ \big((\beta)_n, (\gamma')_n\big) \in \Phi_1^0 \wedge \neg B = \textstyle\bigcup_{n\in\omega}\ C_{(\gamma')_n}^{\omega^\omega} \\ &\Leftrightarrow \exists (\beta,\gamma) \in \Phi_1^1\ \ C_\gamma^{\omega^\omega} = B. \end{aligned}$$

Assume now that the result is proved for $1 \leq \eta < \xi \geq 2$. We get

$$\begin{aligned} B \in \mathbf{\Pi}_\xi^0 &\Leftrightarrow \exists (B_n)_{n\in\omega} \in (\mathbf{\Pi}_{<\xi}^0)^\omega\ \ \neg B = \textstyle\bigcup_{n\in\omega}\ B_n \\ &\Leftrightarrow \exists \beta, \gamma' \in \omega^\omega\ \ \forall n \in \omega\ \ \big((\beta)_n, (\gamma')_n\big) \in \Phi_1^{<\xi} \wedge \neg B = \textstyle\bigcup_{n\in\omega}\ C_{(\gamma')_n}^{\omega^\omega} \\ &\Leftrightarrow \exists (\beta,\gamma) \in \Phi_1^\xi\ \ C_\gamma^{\omega^\omega} = B. \end{aligned}$$

This finishes the proof. $\square$

We now define a Π^1_1 coding of $\mathcal{D}$ (recall Definition 5.1.2).

Notation. We define an inductive operator Λ over ω^ω as follows:

$$\begin{aligned}\Lambda(D):=D\cup\big\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge|(\alpha)_n|=0\big\}\cup\\ \big\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge(\alpha)_0=(\alpha)_2\wedge|(\alpha)_1|=1\wedge<(\alpha)_{2+j}>\in D\big\}\cup\\ \big\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge|(\alpha)_0|\geq 1\wedge|(\alpha)_1|=2\wedge\\ \forall p\in\omega\ <(\alpha)_{2+<p,q>}>\in D\wedge\big(|(\alpha)_{2+<p,0>}|\geq|(\alpha)_0|\vee|(\alpha)_{2+<p,0>}|=0\big)\big\}.\end{aligned}$$

Then Λ is a Π^1_1 monotone inductive operator, by 4A.2 in [M].

By 7C.1 in [M] we get $\Lambda^\infty:=\bigcup_\xi\ \Lambda^\xi=\Lambda(\Lambda^\infty)=\bigcap\{D\subseteq\omega^\omega\mid\Lambda(D)\subseteq D\}$. An easy induction on ξ shows that $\Lambda^\infty\subseteq\big\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\big\}$, so that the coding function c, partially defined by $c(\alpha):=\big(|(\alpha)_n|\big)_{n\in\omega}$, is defined on Λ^∞.

Lemma 6.2 *The set Λ^∞ is a Π^1_1 coding of $\mathcal{D}$, which means that $\Lambda^\infty\in\Pi^1_1(\omega^\omega)$ and $c[\Lambda^\infty]=\mathcal{D}$.*

Proof. We first prove that $\Lambda^\infty\in\Pi^1_1(\omega^\omega)$ (see 7C in [M] for that). We define a set relation $\varphi(\alpha,D)$ on ω^ω by $\varphi(\alpha,D)\Leftrightarrow\alpha\in\Lambda(D)$. As Λ is monotone, φ is operative. If $Q\in\Pi^1_1(Z\times\omega^\omega)$, then the relation $\varphi(\alpha,\{\beta\in\omega^\omega\mid(z,\beta)\in Q\})$ is in Π^1_1. Thus φ is Π^1_1 on Π^1_1. By 7C.8 in [M], $\varphi^\infty(\alpha)$ is in Π^1_1 and $\Lambda^\infty\in\Pi^1_1(\omega^\omega)$.

Let $\beta_\varepsilon\in\text{WO}$ such that $|\beta_\varepsilon|=\varepsilon$, for $\varepsilon\in 3$. Then $<\beta_0\mid n\in\omega>\in\Lambda^0\subseteq\Lambda^\infty$, so that $0^\infty\in c[\Lambda^\infty]$. Let $u^*\in c[\Lambda^\infty]$, $\alpha^*\in\Lambda^\infty$ with $u^*=c(\alpha^*)$. Then $<(\alpha^*)_0,\beta_1,(\alpha^*)_0,(\alpha^*)_1,\ldots>\in\Lambda(\Lambda^\infty)=\Lambda^\infty$, so that $u^*(0)1u^*=c\big(<(\alpha^*)_0,\beta_1,(\alpha^*)_0,(\alpha^*)_1,\ldots>\big)\in c[\Lambda^\infty]$.

Now let $\xi\geq 1$, $u_p\in c[\Lambda^\infty]$ such that $u_p(0)\geq\xi$ or $u_p(0)=0$, for each $p\in\omega$. Choose $\alpha\in\text{WO}$ with $|\alpha|=\xi$, and $\alpha^p\in\Lambda^\infty$ with $u_p=c(\alpha^p)$. Then $<\alpha,\beta_2,(\alpha^{(0)_0})_{(0)_1},(\alpha^{(1)_0})_{(1)_1},\ldots>\in\Lambda(\Lambda^\infty)=\Lambda^\infty$, so that $\xi 2<u_p>=c\big(<\alpha,\beta_2,(\alpha^{(0)_0})_{(0)_1},(\alpha^{(1)_0})_{(1)_1},\ldots>\big)\in c[\Lambda^\infty]$. Thus $\mathcal{D}\subseteq c[\Lambda^\infty]$.

Assume now that $\tilde{\mathcal{D}}\subseteq\omega_1^\omega$ satisfies the following properties:
(a) $0^\infty\in\tilde{\mathcal{D}}$.
(b) $u^*\in\tilde{\mathcal{D}}\Rightarrow u^*(0)1u^*\in\tilde{\mathcal{D}}$.
(c) $\Big(\xi\geq 1\wedge\forall p\in\omega\ \big(u_p\in\tilde{\mathcal{D}}\wedge(u_p(0)\geq\xi\vee u_p(0)=0)\big)\Big)\Rightarrow\xi 2<u_p>\in\tilde{\mathcal{D}}$.

We set $D:=\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge c(\alpha)\in\tilde{\mathcal{D}}\}$. It remains to see that $\Lambda(D)\subseteq D$. Indeed, this will imply that $\Lambda^\infty\subseteq D$, $c[\Lambda^\infty]\subseteq c[D]\subseteq\tilde{\mathcal{D}}$ and $c[\Lambda^\infty]\subseteq\mathcal{D}$.

As $0^\infty\in\tilde{\mathcal{D}}$ we get $\big\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge|(\alpha)_n|=0\big\}\subseteq D$. Assume that $(\alpha)_n\in\text{WO}$ for each $n\in\omega$, that $(\alpha)_0=(\alpha)_2$, $|(\alpha)_1|=1$ and $<(\alpha)_{2+j}>\in D$. Then $u^*:=\big(|(\alpha)_{2+j}|\big)\in\tilde{\mathcal{D}}$, and $|(\alpha)_2|1u^*\in\tilde{\mathcal{D}}$. Thus $c(\alpha)\in\tilde{\mathcal{D}}$ and $\alpha\in D$.

Assume now that $(\alpha)_n\in\text{WO}$ for each $n\in\omega$, $|(\alpha)_0|\geq 1$, $|(\alpha)_1|=2$, $<(\alpha)_{2+<p,q>}>\in D$, and $|(\alpha)_{2+<p,0>}|\geq|(\alpha)_0|$ or $|(\alpha)_{2+<p,0>}|=0$ for each $p\in\omega$. We set $\xi:=|(\alpha)_0|$. Then we have $u_p:=\big(|(\alpha)_{2+<p,q>}|\big)\in\tilde{\mathcal{D}}$, and $\xi 2<u_p>\in\tilde{\mathcal{D}}$. Thus $c(\alpha)\in\tilde{\mathcal{D}}$ and $\alpha\in D$. $\square$

Note that just like Definition 5.1.2, the definition of Λ is cut into three cases, that we will meet again later on: $|(\alpha)_1|=0$ (or, equivalently, $|(\alpha)_n|=0$ for each integer n), $|(\alpha)_1|=1$ or $|(\alpha)_1|=2$.

Even if "$u\in\mathcal{D}$" is the least relation satisfying some conditions, some simplifications are possible. For example, $\mathbf{\Gamma}_{01010^\infty}=\mathbf{\Gamma}_{0^\infty}$. Some other simplifications are possible, and some of them will simplify the notation later on. This will lead to the notion of a normalized code of a description. To define it, we need to associate a tree to a code of a description. The idea is to describe the construction of a set in $\mathbf{\Gamma}_u$ using simpler and simpler sets, until we get the simplest set, namely the empty set. More specifically, we define $\mathfrak{T}:\Lambda^\infty\to\{\text{trees on }\omega\times\Lambda^\infty\}$ as follows. Let $\alpha\in\Lambda^\xi\setminus\Lambda^{<\xi}$. We set

$$\mathfrak{T}(\alpha):=\begin{cases}\{\emptyset\}\cup\{<(0,\alpha)>\}\text{ if }|(\alpha)_1|=0,\\[2ex] \{\emptyset\}\cup\big\{(0,\alpha)^\frown s\mid s\in\mathfrak{T}(<(\alpha)_{2+j}>)\big\}\text{ if }|(\alpha)_1|=1,\\[2ex] \{\emptyset\}\cup\big\{(0,\alpha)^\frown s\mid s\in\mathfrak{T}(<(\alpha)_{2+<0,q>}>)\big\}\cup\\ \qquad\bigcup_{p\geq 1}\ \big\{(p,\alpha)^\frown s\mid s\in\mathfrak{T}(<(\alpha)_{2+<(p)_0+1,q>}>)\big\}\text{ if }|(\alpha)_1|=2.\end{cases}$$

An easy induction on η shows that $\mathfrak{T}(\alpha)$ is always a countable well-founded tree (the first coordinate of (p,α) ensures the well-foundedness). A sequence $s\in\mathfrak{T}(\alpha)$ is said to be $maximal$ if $s\subseteq t\in\mathfrak{T}(\alpha)$ implies that $s=t$. Note that $\big|\big(s_1(|s|-1)\big)_1\big|=0$ if s is maximal. We denote by $\mathcal{M}_\alpha$ the set of maximal sequences in $\mathfrak{T}(\alpha)$.

Definition 6.3 *We say that $\alpha\in\Lambda^\infty$ is $normalized$ if the following holds:*

$$\big(s\in\mathcal{M}_\alpha\wedge i<|s|\wedge\big|\big(s_1(i)\big)_1\big|=1\big)\Rightarrow i=|s|-2.$$

This means that in a maximal sequence s of $\mathfrak{T}(\alpha)$, $\big|\big(s_1(i)\big)_1\big|$ is 2, then possibly 1 once, and finally 0 once. The next lemma says that we can always assume that α is normalized. It is based on the fact that $\check{S_\xi}(\mathbf{\Gamma},\mathbf{\Gamma}')=S_\xi(\check{\mathbf{\Gamma}},\check{\mathbf{\Gamma}}')$.

Lemma 6.4 *Let $\alpha\in\Lambda^\infty$. Then there is $\alpha'\in\Lambda^\infty$ normalized with $(\alpha')_0=(\alpha)_0$ and $\mathbf{\Gamma}_{c(\alpha')}=\mathbf{\Gamma}_{c(\alpha)}$.*

Proof. Assume that $\alpha\in\Lambda^\xi\setminus\Lambda^{<\xi}$. We argue by induction on ξ.

Case 1. $|(\alpha)_1|=0$.

We just set $\alpha':=\alpha$ since $\big|\big(s_1(i)\big)_1\big|=0$.

Case 2. $|(\alpha)_1|=1$.

• We first define $N:\Lambda^\infty\to\Lambda^\infty$ as follows. We ensure that $\big(N(\beta)\big)_0=(\beta)_0$ and $\mathbf{\Gamma}_{c(N(\beta))}=\check{\mathbf{\Gamma}}_{c(\beta)}$. Let $\beta_1\in$WO with $|\beta_1|=1$. We set

$$N(\beta):=\begin{cases}<(\beta)_0,\beta_1,(\beta)_0,(\beta)_1,(\beta)_2,...>\ \text{ if }|(\beta)_1|=0,\\ <(\beta)_{2+j}>\ \text{ if }|(\beta)_1|=1,\\ <(\beta)_0,(\beta)_1,\bigg(\Big(N\big(<(\beta)_{2+<(i-2)_0,q>}>\big)\Big)_{(i-2)_1}\bigg)_{i\geq 2}>\ \text{ if }|(\beta)_1|=2,\end{cases}$$

and one easily checks that N is defined and suitable.

• As $<(\alpha)_{2+j}>\in\Lambda^{<\xi}$, the induction assumption gives $\alpha''\in\Lambda^\infty$ normalized satisfying the equalities $(\alpha'')_0=(\alpha)_2=(\alpha)_0$ and $\mathbf{\Gamma}_{c(\alpha'')}=\mathbf{\Gamma}_{c(<(\alpha)_{2+j}>)}$. In particular,

$$\mathbf{\Gamma}_{c(\alpha)}=\check{\mathbf{\Gamma}}_{c(<(\alpha)_{2+j}>)}=\check{\mathbf{\Gamma}}_{c(\alpha'')}=\mathbf{\Gamma}_{c(N(\alpha''))}.$$

So we have to find $\alpha'\in\Lambda^\infty$ normalized with $(\alpha')_0=(\alpha'')_0$ and $\mathbf{\Gamma}_{c(\alpha')}=\mathbf{\Gamma}_{c(N(\alpha''))}$. Assume that $\alpha''\in\Lambda^\eta\setminus\Lambda^{<\eta}$. We argue by induction on η.

Subcase 1. $|(\alpha'')_1|\leq 1$.

We just set $\alpha':=N(\alpha'')$.

Subcase 2. $|(\alpha'')_1|=2$.

Note that $<(\alpha'')_{2+<p,q>}>$ is normalized since $(0,\alpha'')^\frown s\in\mathcal{M}_{\alpha''}$ (resp., $(p,\alpha'')^\frown s\in\mathcal{M}_{\alpha''}$) if $s\in\mathcal{M}_{(\alpha'')_{2+<0,q>}}$ (resp., $s\in\mathcal{M}_{(\alpha'')_{2+<(p)_0+1,q>}}$ and $p\geq 1$). The induction assumption gives $<(\alpha')_{2+<p,q>}>\in\Lambda^\infty$ normalized with $(\alpha')_{2+<p,0>}=(\alpha'')_{2+<p,0>}$ and

$$\mathbf{\Gamma}_{c(<(\alpha')_{2+<p,q>}>)}=\mathbf{\Gamma}_{c(N(<(\alpha'')_{2+<p,q>}>))}.$$

We set $(\alpha')_i:=(\alpha'')_i$ if $i\in 2$ and we are done.

Case 3. $|(\alpha)_1|=2$.

The induction assumption gives $<(\alpha')_{2+<p,q>}>\in\Lambda^\infty$ normalized satisfying the equalities $(\alpha')_{2+<p,0>}=(\alpha)_{2+<p,0>}$ and $\mathbf{\Gamma}_{c(<(\alpha')_{2+<p,q>}>)}=\mathbf{\Gamma}_{c(<(\alpha)_{2+<p,q>}>)}$. We set $(\alpha')_i:=(\alpha)_i$ if $i\in 2$ and we are done. $\square$

Using Φ_1, we will now code the non self-dual Wadge classes of Borel sets, and define an operator Υ_1 on $(\omega^\omega)^3$ to do it. We set

$$\begin{array}{r}\Upsilon_1(A):=A\cup\Big\{(\alpha,\beta,\gamma)\in(\omega^\omega)^2\times W^{\omega^\omega}\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge\hfill\\
\Big(\forall n\in\omega\ \ |(\alpha)_n|=0\wedge\beta(0)=0\wedge C^{\omega^\omega}_\gamma=\emptyset\Big)\vee\\
\Big(|(\alpha)_1|=1\wedge(\alpha)_0=(\alpha)_2\wedge\beta(0)=1\wedge\\
\exists\gamma'\in\omega^\omega\ \ (<(\alpha)_{2+j}>,\beta^*,\gamma')\in A\wedge C^{\omega^\omega}_\gamma=\neg C^{\omega^\omega}_{\gamma'}\Big)\vee\\
\Big(|(\alpha)_1|=2\wedge|(\alpha)_0|\geq 1\wedge\forall p\in\omega\ \ \big(|(\alpha)_{2+<p,0>}|\geq|(\alpha)_0|\vee|(\alpha)_{2+<p,0>}|=0\big)\wedge\\
\beta(0)=2\wedge\exists\gamma'\in\omega^\omega\ \ (<(\alpha)_{2+<0,q>}>,(\beta^*)_0,(\gamma')_0)\in A\wedge\\
\forall p\geq 1\ \ \big(<(\alpha)_{2+<(p)_0+1,q>}>,\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\big)\in A\wedge\big(\big((\beta^*)_p\big)_1,\big((\gamma')_p\big)_1\big)\in\Phi_1^{|(\alpha)_0|}\wedge\\
\forall p\neq q\geq 1\ \ C^{\omega^\omega}_{((\gamma')_p)_1}\cup C^{\omega^\omega}_{((\gamma')_q)_1}=\omega^\omega\wedge\\
C^{\omega^\omega}_\gamma=\bigcup_{p\geq 1}\,(C^{\omega^\omega}_{((\gamma')_p)_0}\setminus C^{\omega^\omega}_{((\gamma')_p)_1})\cup(C^{\omega^\omega}_{(\gamma')_0}\cap\bigcap_{p\geq 1}\ C^{\omega^\omega}_{((\gamma')_p)_1})\Big)\Big\}.\end{array}$$

Lemma 6.5 *Let ξ be an ordinal.*

(a) Assume that $(\alpha,\beta,\gamma)\in\Upsilon_1^\xi$. Then $\alpha\in\Lambda^\xi$.

(b) Let $\alpha\in\Lambda^\xi$ and $B\subseteq\omega^\omega$. Then $B\in\mathbf{\Gamma}_{c(\alpha)}$ if and only if there are $\beta,\gamma\in\omega^\omega$ such that $(\alpha,\beta,\gamma)\in\Upsilon_1^\xi$ and $C_\gamma^{\omega^\omega}=B$.

Proof. (a) We argue by induction on ξ. So let $\alpha\in\Upsilon_1^\xi\setminus\Upsilon_1^{<\xi}$. We may assume that $|(\alpha)_1|\geq 1$. If $|(\alpha)_1|=1$, then $(<(\alpha)_{2+j}>,\beta^*,\gamma')\in\Upsilon_1^{<\xi}$ for some γ' and $<(\alpha)_{2+j}>\in\Lambda^{<\xi}$ by induction assumption, so we are done. If $|(\alpha)_1|=2$, then

$$(<(\alpha)_{2+<0,q>}>,(\beta^*)_0,(\gamma')_0),\big(<(\alpha)_{2+<(p)_0+1,q>}>,\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\big)\in\Upsilon_1^{<\xi}$$

for some γ' and $<(\alpha)_{2+<p,q>}>\in\Lambda^{<\xi}$ by induction assumption for each integer p.

(b) $\Rightarrow$ We argue by induction on ξ, and we may assume that $\alpha\notin\Lambda^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Note that $c(\alpha)=0^\infty$ and $B=\emptyset$. We set $\beta:=0^\infty$, and we choose $\gamma\in W^{\omega^\omega}$ with $C_\gamma=\emptyset$. Then $(\alpha,\beta,\gamma)\in\Upsilon_1^0\subseteq\Upsilon_1^\xi$.

Case 2. $|(\alpha)_1|=1$.

Note that $<(\alpha)_{2+j}>\in\Lambda^{<\xi}$, and $\neg B\in\mathbf{\Gamma}_{c(<(\alpha)_{2+j}>)}$. By induction assumption we get $\beta',\gamma'\in\omega^\omega$ such that $(<(\alpha)_{2+j}>,\beta',\gamma')\in\Upsilon_1^{<\xi}$ and $C_{\gamma'}^{\omega^\omega}=\neg B$. We set $\beta:=1\beta'$ and we choose $\gamma\in W^{\omega^\omega}$ with $C_\gamma^{\omega^\omega}=\neg C_{\gamma'}^{\omega^\omega}$.

Case 3. $|(\alpha)_1|=2$.

Note that $<(\alpha)_{2+<p,q>}>\in\Lambda^{<\xi}$ for each integer p. We can write

$$B=\bigcup_{p\geq 1}(A_p\cap C_p)\cup(B'\setminus\bigcup_{p\geq 1}C_p),$$

where $(C_p)_{p\geq 1}$ is a sequence of pairwise disjoint $\mathbf{\Sigma}^0_{|(\alpha)_0|}$ sets, $B'\in\mathbf{\Gamma}_{c(<(\alpha)_{2+<0,q>}>)}$ and

$$A_p\in\mathbf{\Gamma}_{c(<(\alpha)_{2+<(p)_0+1,q>}>)}.$$

Lemma 6.1 gives $\big(\big((\beta^*)_p\big)_1,\big((\gamma')_p\big)_1\big)\in\Phi_1^{|(\alpha)_0|}$ such that $C_{((\gamma')_p)_1}^{\omega^\omega}=\neg C_p$. The induction assumption gives $(\beta^*)_0,(\gamma')_0\in\omega^\omega$ such that $(<(\alpha)_{2+<0,q>}>,(\beta^*)_0,(\gamma')_0)\in\Upsilon_1^{<\xi}$ and $C_{(\gamma')_0}^{\omega^\omega}=B'$, and $\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\in\omega^\omega$ such that $\big(<(\alpha)_{2+<(p)_0+1,q>}>,\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\big)\in\Upsilon_1^{<\xi}$ and $C_{((\gamma')_p)_0}^{\omega^\omega}=A_p$. We set $\beta(0):=2$ and we choose $\gamma\in W^{\omega^\omega}$ with

$$C_\gamma^{\omega^\omega}=\bigcup_{p\geq 1}(C_{((\gamma')_p)_0}^{\omega^\omega}\setminus C_{((\gamma')_p)_1}^{\omega^\omega})\cup(C_{(\gamma')_0}^{\omega^\omega}\cap\bigcap_{p\geq 1}C_{((\gamma')_p)_1}^{\omega^\omega}).$$

$\Leftarrow$ We argue by induction on ξ, and we may assume that $(\alpha,\beta,\gamma)\notin\Upsilon_1^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Note that $B=C^{\omega^\omega}_\gamma=\emptyset\in\mathbf{\Gamma}_{0^\infty}=\mathbf{\Gamma}_{c(\alpha)}$.

Case 2. $|(\alpha)_1|=1$.

Note that there is γ' such that $(<(\alpha)_{2+j}>,\beta^*,\gamma')\in\Upsilon_1^{<\xi}$ and $C^{\omega^\omega}_\gamma=\neg C^{\omega^\omega}_{\gamma'}$, which implies that $B\in\check{\mathbf{\Gamma}}_{c(<(\alpha)_{2+j}>)}=\mathbf{\Gamma}_{c(\alpha)}$.

Case 3. $|(\alpha)_1|=2$.

We get γ' since $(\alpha,\beta,\gamma)\in\Upsilon_1^\xi$. As

$$(<(\alpha)_{2+<0,q>}>,(\beta^*)_0,(\gamma')_0),\big(<(\alpha)_{2+<(p)_0+1,q>}>,\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\big)\in\Upsilon_1^{<\xi}$$

we get $C^{\omega^\omega}_{(\gamma')_0}\in\mathbf{\Gamma}_{c(<(\alpha)_{2+<0,q>}>)}$ and $C^{\omega^\omega}_{((\gamma')_p)_0}\in\mathbf{\Gamma}_{c(<(\alpha)_{2+<(p)_0+1,q>}>)}$, by induction assumption. As $\big(\big((\beta^*)_p\big)_1,\big((\gamma')_p\big)_1\big)\in\Phi_1^{|(\alpha)_0|}$, we get $C^{\omega^\omega}_{((\gamma')_p)_1}\in\mathbf{\Pi}^0_{|(\alpha)_0|}$ by Lemma 6.1. This implies that

$$B\in S_{|(\alpha)_0|}\big(\bigcup_{p\geq 1}\mathbf{\Gamma}_{c(<(\alpha)_{2+<p,q>}>)},\mathbf{\Gamma}_{c(<(\alpha)_{2+<0,q>}>)}\big)=\mathbf{\Gamma}_{c(\alpha)}.$$

This finishes the proof. $\square$

Remark. We will also consider the operator Υ defined just like Υ_1, except that
- We replace $(W^{\omega^\omega},C^{\omega^\omega})$ with (W,C) (we work in $(\omega^\omega)^d$ instead of ω^ω).
- We replace the condition of the form $(\tilde\beta,\tilde\gamma)\in\Phi_1^{|(\alpha)_0|}$ with $\big((\alpha)_0,\tilde\beta,\tilde\gamma\big)\in Q$ (see the remark at the end of Section 4 for the definition of Q).
- We ask β,γ,γ' to be $\varDelta^1_1(\alpha)$, so that Υ is a $\varPi^1_1$ monotone inductive operator.

To prove Theorem 1.8, we will consider some tuples $\vec v:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)$, where $\alpha\in\Lambda^\infty$. We will inductively define them through an inductive operator over $(\omega^\omega)^6$ called Θ. The definition of Θ is in the spirit of that of Υ_1, and is cut into three cases, depending on the value of $|(\alpha)_1|$. As the definition of Θ is long and technical, we give first some more informal explanations about its meaning. We will have $\vec v\in\Theta^\infty$. So there is an ordinal ξ such that $\vec v\in\Theta^\xi$.

- $\alpha\in\Lambda^\xi$ is a (normalized in practice) code for a description $u=c(\alpha)$.

- $a_0,a_1\in\varDelta^1_1(\alpha)$ are codes for a pair of disjoint analytic subsets of $(\omega^\omega)^d$. Using the good universal set $\mathcal{U}$ for $\varPi^1_1$ defined after the proof of Theorem 4.2.2, at the end of Section 4, we will actually code the complement of these analytic sets, so that we will set $A_i:=\neg\mathcal{U}_{a_i}$ for $i\in 2$.

- Similarly, $\underline{a}_0,\underline{a}_1\in\varDelta^1_1(\alpha)$ are codes for a pair of disjoint analytic subsets of $(\omega^\omega)^d$. In fact, we will have $\underline{A}_i:=\neg\mathcal{U}_{\underline{a}_i}\subseteq A_i$. These codes will be used to build r, and $\underline{a}_0,\underline{a}_1,r$ will be completely determined by (α,a_0,a_1). So one should think that $\underline{a}_i=\underline{a}_i(\alpha,a_0,a_1)\simeq\underline{a}_i(u,a_0,a_1)$, $r=r(\alpha,a_0,a_1)\simeq r(u,a_0,a_1)$. We need the following lemma to specify their meaning.

Lemma 6.6 *There is a recursive map* $f_a:(\omega^\omega)^2\rightarrow\omega^\omega$ *such that* $\mathcal{U}_{f_a(\alpha,r)}=\mathcal{U}_{(r)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\neg\mathcal{U}_{(r)_p}}^{\tau_{|\alpha|}}$ *if* $\alpha\in\Delta^1_1\cap$ *WO and* $|\alpha|\geq 1$.

Proof. Note first that $P:=\{(\beta,\vec{\delta})\in\omega^\omega\times(\omega^\omega)^d\mid(\beta)_0\in\Delta^1_1\cap\text{WO}\wedge|(\beta)_0|\geq 1\wedge$

$$\vec{\delta}\in\mathcal{U}_{((\beta)_1)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\neg\mathcal{U}_{((\beta)_1)_p}}^{\tau_{|(\beta)_0|}}\}$$

is a Π^1_1 set, by the remark at the end of Section 4 defining R. This gives $\gamma\in\omega^\omega$ recursive with $P=\mathcal{U}_\gamma^{\omega^\omega\times(\omega^\omega)^d}$. Let $\alpha\in\Delta^1_1\cap$ WO with $|\alpha|\geq 1$, and $r\in\omega^\omega$. We have

$$\begin{array}{ll}\vec{\delta}\in\mathcal{U}_{(r)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\neg\mathcal{U}_{(r)_p}}^{\tau_{|\alpha|}} & \Leftrightarrow(<\alpha,r,r,...>,\vec{\delta}\,)\in P\\ & \Leftrightarrow(\gamma,<\alpha,r,r,...>,\vec{\delta}\,)\in\mathcal{U}^{\omega^\omega\times(\omega^\omega)^d}\\ & \Leftrightarrow\big(S(\gamma,<\alpha,r,r,...>),\vec{\delta}\,\big)\in\mathcal{U}\end{array}$$

We just have to set $f_a(\alpha,r):=S(\gamma,<\alpha,r,r,...>)$. □

The following will hold:

$\circ$ If $u=0^\infty$ or $u=\xi 1u^*$, then $\underline{a}_i=\underline{a}_i(\alpha,a_0,a_1)=\underline{a}_i(u,a_0,a_1)=a_i$.

$\circ$ If $u=\xi 2<u_p>$, then there will be $a'_0,a'_1,r'\in\Delta^1_1(\alpha)$ such that, for each $p\geq 1$,

$$\big(<(\alpha)_{2+<(p)_0+1,q>}>,a_0,a_1,(a'_0)_p,(a'_1)_p,(r')_p\big)\in\Theta^{<\xi}.$$

We will have $\underline{a}_i=\underline{a}_i(u,a_0,a_1)=f_a\big((\alpha)_0,<a_i,(r')_1,(r')_2,...>\big)$, and $(r')_p=r(u_{(p)_0+1},a_0,a_1)$ if $p\geq 1$. In particular, $\underline{A}_i=A_i\cap\bigcap_{p\geq 1}\ \overline{\neg\mathcal{U}_{r(u_{(p)_0+1},a_0,a_1)}}^{\tau_\xi}$.

- $r\in\Delta^1_1(\alpha)$ is a code for an analytic subset of $(\omega^\omega)^d$ playing the role that $\overline{A_0}^{\tau_\xi}\cap A_1$ played in Theorem 4.2.2. In other words, the emptyness of this analytic set is equivalent to the possibility of separating A_0 from A_1 by a pot($\mathbf{\Gamma}_u$) set. Here again, using $\mathcal{U}$, we will actually code the complement of this analytic set: $\neg\mathcal{U}_r$ is an analytic subset of $(\omega^\omega)^d$. In particular,

$\circ$ If $u=0^\infty$, then $r=r(\alpha,a_0,a_1)=r(u,a_0,a_1)=a_1$.

$\circ$ If $u=\xi 1u^*$, then $r=r(\alpha,a_0,a_1)=r(u,a_0,a_1)=a_0$.

$\circ$ If $u=\xi 2<u_p>$, then we there will be $a''_0,a''_1\in\Delta^1_1(\alpha)$ such that

$$(<(\alpha)_{2+<0,q>}>,\underline{a}_0,\underline{a}_1,a''_0,a''_1,r)\in\Theta^{<\xi}.$$

In particular, $r(u,a_0,a_1)=r(u_0,\underline{a}_0,\underline{a}_1)=r\big(u_0,\underline{a}_0(u,a_0,a_1),\underline{a}_1(u,a_0,a_1)\big)$. We are now ready to define Θ (recall the remark at the end of Section 4 defining Q).

The operator Θ is defined as follows (recall the definition of Λ):

$$\Theta(A):=A\cup\Bigg\{(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\big(\omega^\omega\cap\Delta^1_1(\alpha)\big)^6\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge$$

$$\Big(\forall n\in\omega\ \ |(\alpha)_n|=0\wedge\mathcal{U}_{a_0}\cup\mathcal{U}_{a_1}=(\omega^\omega)^d\wedge(\underline{a}_0,\underline{a}_1)=(a_0,a_1)\wedge r=a_1\Big)\vee$$

$$\Big(|(\alpha)_1|=1\wedge(\alpha)_0=(\alpha)_2\wedge(<(\alpha)_{2+j}>,a_0,a_1,\underline{a}_0,\underline{a}_1,a_1)\in A\wedge r=a_0\Big)\vee$$

$$\Big(|(\alpha)_1|=2\wedge|(\alpha)_0|\geq 1\wedge\forall p\in\omega\ \ \big(|(\alpha)_{2+<p,0>}|\geq|(\alpha)_0|\vee|(\alpha)_{2+<p,0>}|=0\big)\wedge$$

$$\exists a'_0,a'_1,r'\in\Delta^1_1(\alpha)\ \ \big(<(\alpha)_{2+<0,q>}>,a_0,a_1,(a'_0)_0,(a'_1)_0,(r')_0\big)\in A\wedge$$

$$\forall p\geq 1\ \ \big(<(\alpha)_{2+<(p)_0+1,q>}>,a_0,a_1,(a'_0)_p,(a'_1)_p,(r')_p\big)\in A\wedge$$

$$\forall i\in 2\ \ \underline{a}_i=f_a\big((\alpha)_0,<a_i,(r')_1,(r')_2,...>\big)\wedge$$

$$\exists a''_0,a''_1\in\Delta^1_1(\alpha)\ \ (<(\alpha)_{2+<0,q>}>,\underline{a}_0,\underline{a}_1,a''_0,a''_1,r)\in A\Big)\Bigg\}.$$

Then Θ is a Π^1_1 monotone inductive operator.

Remark. Let ξ be an ordinal, and $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\xi$. Then an induction on ξ shows the following properties:

- $\neg\mathcal{U}_{a_0}\cap\neg\mathcal{U}_{a_1}=\emptyset$.
- $\neg\mathcal{U}_{\underline{a}_i}\subseteq\neg\mathcal{U}_{a_i}$ for each $i\in 2$. In particular, $\neg\mathcal{U}_{\underline{a}_0}\cap\neg\mathcal{U}_{\underline{a}_1}=\emptyset$.
- $\underline{a}_0,\underline{a}_1,r$ are completely determined by (α,a_0,a_1).
- If $\neg\mathcal{U}_{a_i}\subseteq\neg\mathcal{U}_{b_i}$ for each $i\in 2$, then $\neg\mathcal{U}_{\underline{a}_i}\subseteq\neg\mathcal{U}_{\underline{b}_i}$ for each $i\in 2$ and $\neg\mathcal{U}_{r(\alpha,a_0,a_1)}\subseteq\neg\mathcal{U}_{r(\alpha,b_0,b_1)}$.
- There is $i\in 2$ such that $\neg\mathcal{U}_r\subseteq\neg\mathcal{U}_{a_i}$.

Lemma 6.7 *(a) Let ξ be an ordinal, $\alpha\in\Delta^1_1$, and $(\alpha,\beta,\gamma)\in\Upsilon^\xi$. Then $\alpha\in\Lambda^\xi$ and the set C_γ is in $\Delta^1_1\cap\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$.*
(b) Let $\alpha\in\Delta^1_1\cap\Lambda^\infty$ normalized, $a_0,a_1\in\Delta^1_1$ with $A_0\cap A_1=\emptyset$. Then there are $\underline{a}_0,\underline{a}_1,r\in\omega^\omega$ such that $(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$.

Proof. (a) We argue as in the proof of Lemmas 6.5.(a) and 6.5.(b)$\Leftarrow$. The only thing to notice is that in the case $|(\alpha)_1|=2$, $\big((\alpha)_0,\big((\beta^*)_p\big)_1,\big((\gamma')_p\big)_1\big)\in Q$. Proposition 2.2, Lemma 2.3 and Theorem 3.1 give a tree T_d with Δ^1_1 suitable levels and $S\in\mathbf{\Sigma}^0_{|(\alpha)_0|}(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_{|(\alpha)_0|}$) set. As $\alpha\in\Delta^1_1$, $|(\alpha)_0|<\omega_1^{\text{CK}}$ and Theorem 4.2.2 implies that $C_{((\gamma')_p)_1}$ is in $\mathbf{\Pi}^0_{|(\alpha)_0|}(\tau_1)$. Thus $C_\gamma\in\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$.

(b) Let ξ be an ordinal with $\alpha\in\Lambda^\xi$. Here again we argue by induction on ξ. So assume that $\alpha\notin\Lambda^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Let $\underline{a}_i:=a_i$ and $r:=a_1$. Then $(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^0\subseteq\Theta^\infty$.

Case 2. $|(\alpha)_1| = 1$.

As $<(\alpha)_{2+j}>\in\Lambda^{<\xi}$ we get, by induction assumption, $(\underline{a}_0, \underline{a}_1, r')$ with

$$(<(\alpha)_{2+j}>, a_0, a_1, \underline{a}_0, \underline{a}_1, r')\in\Theta^\infty.$$

As α is normalized we get $|(\alpha)_{2+j}| = 0$ for each j, and $r' = a_1$. We set $r := a_0$. Then

$$(\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r)\in\Theta(\Theta^\infty) = \Theta^\infty.$$

Case 3. $|(\alpha)_1| = 2$.

As $<(\alpha)_{2+<p,q>}>\in\Lambda^{<\xi}$ we get, by induction assumption, (a_0^p, a_1^p, r'_p) with

$$\big(<(\alpha)_{2+<0,q>}>, a_0, a_1, a_0^0, a_1^0, r'_0\big)\in\Theta^\infty,$$

and $(<(\alpha)_{2+<(p)_0+1,q>}>, a_0, a_1, a_0^p, a_1^p, r'_p)\in\Theta^\infty$, for each $p\geq 1$. As in the proof of Lemma 6.2 we see that $\Theta^\infty\in\mathit{\Pi}_1^1$. By $\mathit{\Delta}_1^1$-selection, we may assume that the sequences (a_0^p), (a_1^p) and (r'_p) are $\mathit{\Delta}_1^1$. In particular, there is $a'_i\in\mathit{\Delta}_1^1$ with $(a'_i)_p = a_i^p$. We set $(r')_p := r'_p$, and

$$\underline{a}_i := f_a\big((\alpha)_0, < a_i, (r')_1, (r')_2, ... >\big).$$

The induction assumption gives a''_0, a''_1, r such that $(<(\alpha)_{2+<0,q>}>, \underline{a}_0, \underline{a}_1, a''_0, a''_1, r)\in\Theta^\infty$. We are done since $(\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r)\in\Theta^\infty$. □

The next lemma is the crucial separation lemma announced in the presentation of r.

Lemma 6.8 *Let $\vec{v} := (\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r)\in\Theta^\infty$ with $\alpha\in\mathit{\Delta}_1^1$ normalized and $a_0, a_1\in\mathit{\Delta}_1^1$, Σ in $\mathit{\Sigma}_1^1\big((\omega^\omega)^d\big)$ with $(\neg\mathcal{U}_r)\cap\Sigma = \emptyset$. Then there are $\beta', \gamma'\in\omega^\omega$ such that $(\alpha, \beta', \gamma')\in\Upsilon^\infty$ and $C_{\gamma'}$ separates $A_1\cap\Sigma$ from $A_0\cap\Sigma$. In particular, $A_1\cap\Sigma$ is separable from $A_0\cap\Sigma$ by a $\mathit{\Delta}_1^1\cap\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$ set.*

Proof. The last assertion comes from Lemma 6.7.(a). Let η be an ordinal with $\vec{v}\in\Theta^\eta$. We argue by induction on η. So assume that $\vec{v}\in\Theta^\eta\setminus\Theta^{<\eta}$.

Case 1. $|(\alpha)_1| = 0$.

We set $\beta' := 0^\infty$, and choose $\gamma'\in\mathit{\Delta}_1^1\cap W$ with $C_{\gamma'} = \emptyset$. We are done since $\emptyset = A_1\cap\Sigma$.

Case 2. $|(\alpha)_1| = 1$.

As α is normalized, we get $|(\alpha)_{2+j}| = 0$ for each j. We set $\beta' := 10^\infty$, and choose $\gamma'\in\mathit{\Delta}_1^1\cap W$ with $C_{\gamma'} = (\omega^\omega)^d$. Then $\gamma''\in\mathit{\Delta}_1^1\cap W$ with $C_{\gamma''} = \emptyset$ is a witness for the fact that $(\alpha, \beta', \gamma')\in\Upsilon^\infty$. We are done since $r = a_0$.

Case 3. $|(\alpha)_1| = 2$.

There are $a'_0, a'_1, r'\in\mathit{\Delta}_1^1$ with $\big(<(\alpha)_{2+<(p)_0+1,q>}>, a_0, a_1, (a'_0)_p, (a'_1)_p, (r')_p\big)\in\Theta^{<\eta}$, for each $p\geq 1$, and, for each $i\in 2$, $\underline{a}_i = f_a\big((\alpha)_0, < a_i, (r')_1, (r')_2, ... >\big)$. Moreover, there are $a''_0, a''_1\in\mathit{\Delta}_1^1$ with $(<(\alpha)_{2+<0,q>}>, \underline{a}_0, \underline{a}_1, a''_0, a''_1, r)\in\Theta^{<\eta}$.

By Lemma 6.7.(a), one of the goals is to build $C_{\gamma'}\in\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$. The proof of Lemma 6.7.(a) shows that $\mathbf{\Gamma}_{c(\alpha)} = S_{|(\alpha)_0|}(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{c(<(\alpha)_{2+<p,q>}>)}, \mathbf{\Gamma}_{c(<(\alpha)_{2+<0,q>}>)})$. This means that we want to find sequences $(C_p)_{p\geq 1}$, $(S_p)_{p\geq 1}$ and B such that $C_{\gamma'} = \bigcup_{p\geq 1}\ (S_p\cap C_p)\cup(B\setminus\bigcup_{p\geq 1}\ C_p)$.

- Let us construct B.

The induction assumption gives $\beta''', \gamma''' \in \omega^\omega$ such that $(< (\alpha)_{2+<0,q>} >, \beta''', \gamma''') \in \Upsilon^\infty$ and $C_{\gamma'''}$ separates $\underline{A}_1 \cap \Sigma$ from $\underline{A}_0 \cap \Sigma$. We set $B := C_{\gamma'''}$.

- Let us construct the C_p's.

We set $\xi := |(\alpha)_0|$. Note that $\underline{A}_i = A_i \cap \bigcap_{p\geq 1} \overline{\neg\mathcal{U}_{(r')_p}}^{\tau_\xi}$. This implies that

$$U := \big(C_{\gamma'''} \cap A_0 \cap \Sigma\big) \cup \big(\neg C_{\gamma'''} \cap A_1 \cap \Sigma\big) \subseteq \bigcup_{p\geq 1} \neg\overline{\neg\mathcal{U}_{(r')_p}}^{\tau_\xi}.$$

As in the proof of Lemma 6.6 we see that the relation "$\vec{\delta} \notin \overline{\neg\mathcal{U}_{(r')_p}}^{\tau_{|(\alpha)_0|}}$" is Π^1_1 in $(p, \alpha, r', \vec{\delta})$. By Δ^1_1-selection there is a Δ^1_1-recursive map $f : (\omega^\omega)^d \to \omega$ such that $f(\vec{\delta}) \geq 1$ for each $\vec{\delta} \in (\omega^\omega)^d$ and $\vec{\delta} \notin \overline{\neg\mathcal{U}_{(r')_{f(\vec{\delta})}}}^{\tau_\xi}$ for each $\vec{\delta} \in U$.

In particular, for each $\vec{\delta} \in U$ there is $P \in \Sigma^1_1 \cap \mathbf{\Pi}^0_{<\xi}(\tau_1)$ such that $\vec{\delta} \in P \subseteq \mathcal{U}_{(r')_{f(\vec{\delta})}}$. Now P and $\neg\mathcal{U}_{(r')_{f(\vec{\delta})}}$ are disjoint Σ^1_1 sets, and separable by a $\mathbf{\Pi}^0_{<\xi}(\tau_1)$ set. As $\alpha \in \Delta^1_1$ we get $1 \leq |(\alpha)_0| < \omega_1^{\mathrm{CK}}$. As in the proof of Lemma 6.7.(a) we get T_d and S. By Theorem 4.2.2 we get $(\beta, \gamma) \in (\Delta^1_1 \times \Delta^1_1) \cap V_{<\xi}$ with $P \subseteq C_\gamma \subseteq \mathcal{U}_{(r')_{f(\vec{\delta})}}$.

By Lemma 4.2.3.(2).(a) the relation "(β, γ) is in $(\Delta^1_1 \times \Delta^1_1) \cap V_{<\xi}$" is Π^1_1, so there is a Δ^1_1-recursive map $g : (\omega^\omega)^d \to \omega \times (\omega^\omega \times \omega^\omega)$ such that

$$\forall \vec{\delta} \in U \ \ g_0(\vec{\delta}) = f(\vec{\delta}) \ \text{ and } \ g_1(\vec{\delta}) \in (\Delta^1_1 \times \Delta^1_1) \cap V_{<\xi} \ \text{ and } \ \vec{\delta} \in C_{(g_1(\vec{\delta}))_1} \subseteq \mathcal{U}_{(r')_{f(\vec{\delta})}},$$

by Δ^1_1-selection. In particular, the Σ^1_1 set $g[U]$ is a subset of

$$\big\{\big(p, (\beta, \gamma)\big) \in \omega \times \big((\Delta^1_1 \times \Delta^1_1) \cap V_{<\xi}\big) \mid C_\gamma \subseteq \mathcal{U}_{(r')_p}\big\},$$

which is Π^1_1 and countable. The separation theorem gives $D \in \Delta^1_1$ between these two sets. As D is countable, there are $N, \tilde{\beta}, \tilde{\gamma} \in \Delta^1_1$ with $D = \Big\{\Big(N(q), \big((\tilde{\beta})_q, (\tilde{\gamma})_q\big)\Big) \mid q \in \omega\Big\}$. Now we can define $C_p := \bigcup_{q\in\omega, N(q)=p} C_{(\tilde{\gamma})_q} \setminus (\bigcup_{q'<q} C_{(\tilde{\gamma})_{q'}})$.

- We now study the properties of the C_p's. We can say that

$\circ$ The relation "$\vec{\delta} \in C_p$" is Δ^1_1 in $(p, \vec{\delta})$.

$\circ$ The C_p's are pairwise disjoint.

$\circ$ $C_p \in \mathbf{\Sigma}^0_\xi(\tau_1)$ since $C_{(\tilde{\gamma})_q} \in \mathbf{\Pi}^0_{<\xi}(\tau_1) \subseteq \mathbf{\Delta}^0_\xi(\tau_1)$, by Theorem 4.2.2.

$\circ$ We set $\tilde{C} := \{(p, \vec{\delta}) \in \omega \times (\omega^\omega)^d \mid \exists q \in \omega \ \ N(q) = p \ \text{ and } \ \vec{\delta} \in C_{(\tilde{\gamma})_q}\}$, so that $\tilde{C} \in \Delta^1_1$ and $\tilde{C}_p \in \mathbf{\Sigma}^0_1(\tau_\xi)$ for each $p \geq 1$. We have $C_p \subseteq \tilde{C}_p$.

$\circ$ $\bigcup_{p\geq 1} C_p = \bigcup_{p\geq 1} \tilde{C}_p$.

$\circ$ $\tilde{C}_p$ separates $U \cap f^{-1}(\{p\})$ from $\neg\mathcal{U}_{(r')_p}$. In particular, U is a subset of the Δ^1_1 set $\bigcup_{p\geq 1} C_p$. Moreover, $\bigcap_{p\geq 1} \overline{\neg\mathcal{U}_{(r')_p}}^{\tau_\xi} \subseteq \neg(\bigcup_{p\geq 1} \tilde{C}_p)$.

- The induction assumption gives, for each $p \geq 1$, β^p, γ^p with $(< (\alpha)_{2+<(p)_0+1,q>} >, \beta^p, \gamma^p) \in \Upsilon^\infty$ and C_{γ^p} separates $A_1 \cap \tilde{C}_p$ from $A_0 \cap \tilde{C}_p$. As in the proof of Lemma 6.7.(b) we may assume that the sequences (β^p) and (γ^p) are Δ^1_1. By Δ^1_1-selection again there is a Δ^1_1-recursive map $h:\omega \to \omega^\omega \times \omega^\omega$ such that $h(p) \in (\Delta^1_1 \times \Delta^1_1) \cap V_\xi$ and $C_{h_1(p)} = \neg C_p$ for each $p \geq 1$. We set $\big((\beta'^*)_p\big)_1 := h_0(p)$ and $\big((\overline{\gamma})_p\big)_1 := h_1(p)$, so that $\Big((\alpha)_0, \big((\beta'^*)_p\big)_1, \big((\overline{\gamma})_p\big)_1\Big) \in Q$ for each $p \geq 1$.

We set $\beta'(0) := 2$, $(\beta'^*)_0 := \beta'''$, and $\big((\beta'^*)_p\big)_0 := \beta^p$ if $p \geq 1$, so that β' is completely defined. Similarly, we set $(\overline{\gamma})_0 := \gamma'''$, and $\big((\overline{\gamma})_p\big)_0 := \gamma^p$ if $p \geq 1$. Finally, we choose $\gamma' \in \Delta^1_1 \cap W$ with $C_{\gamma'} = \bigcup_{p \geq 1} (C_{\gamma^p} \setminus C_{h_1(p)}) \cup (C_{(\overline{\gamma})_0} \cap \bigcap_{p \geq 1} C_{h_1(p)})$, so that $(\alpha, \beta', \gamma') \in \Upsilon^\infty$ and $C_{\gamma'}$ separates $A_1 \cap \Sigma$ from $A_0 \cap \Sigma$. $\square$

The next result is the actual (effective) content of Theorem 1.8.(1). It is also the version of Theorem 4.4.1 for the non self-dual Wadge classes of Borel sets. Let $j_d:(d^\omega)^d \to \omega^\omega$ be a continuous embedding (for example we can embed $(d^\omega)^d$ into $(\omega^\omega)^d$ in the obvious way, and then use a bijection between $(\omega^\omega)^d$ and ω^ω).

Theorem 6.9 *Let T_d be a tree with Δ^1_1 suitable levels, α in Δ^1_1 normalized, β, γ in ω^ω such that $(\alpha, \beta, \gamma) \in \Upsilon^\infty_1$, $S := j_d^{-1}(C^{\omega^\omega}_\gamma) \cap \lceil T_d \rceil$, and $a_0, a_1, \underline{a}_0, \underline{a}_1, r \in \omega^\omega$ with $\vec{v} := (\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r) \in \Theta^\infty$. Then one of the following holds:*

(a) $\neg\mathcal{U}_r = \emptyset$.

(b) The inequality $\big((\Pi''_i \lceil T_d \rceil)_{i \in d}, S, \lceil T_d \rceil \setminus S\big) \leq \big((\omega^\omega)_{i \in d}, A_0, A_1\big)$ holds.

Now we can state the version of Theorem 4.2.2 for the non self-dual Wadge classes of Borel sets.

Theorem 6.10 *Let T_d be a tree with Δ^1_1 suitable levels, α in Δ^1_1 normalized, β, γ in ω^ω such that $(\alpha, \beta, \gamma) \in \Upsilon^\infty_1$, $S := j_d^{-1}(C^{\omega^\omega}_\gamma) \cap \lceil T_d \rceil$, and $a_0, a_1, \underline{a}_0, \underline{a}_1, r \in \omega^\omega$ with $\vec{v} := (\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r) \in \Theta^\infty$. We assume that S is not separable from $\lceil T_d \rceil \setminus S$ by a pot$(\check{\mathbf{\Gamma}}_{c(\alpha)})$ set. Then the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot$(\check{\mathbf{\Gamma}}_{c(\alpha)})$ set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1 \cap$ pot$(\check{\mathbf{\Gamma}}_{c(\alpha)})$ set.

(c) $\neg\big(\exists \beta', \gamma' \in \omega^\omega$ such that $(\alpha, \beta', \gamma') \in \Upsilon^\infty$ and $A_1 \subseteq C_{\gamma'} \subseteq \neg A_0\big)$.

(d) The set A_0 is not separable from A_1 by a $\check{\mathbf{\Gamma}}_{c(\alpha)}(\tau_1)$ set.

(e) $\neg\mathcal{U}_r \neq \emptyset$.

(f) The inequality $\big((d^\omega)_{i \in d}, S, \lceil T_d \rceil \setminus S\big) \leq \big((\omega^\omega)_{i \in d}, A_0, A_1\big)$ holds.

Proof. (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) are clear since Δ_{ω^ω} is Polish.

(b) $\Rightarrow$ (c) This comes from Lemma 6.7.(a).

(b) $\Rightarrow$ (e), (c) $\Rightarrow$ (e) and (d) $\Rightarrow$ (e) This comes from Lemma 6.8.

(e) $\Rightarrow$ (f) This comes from Theorem 6.9 (as $\Pi''_i \lceil T_d \rceil$ is compact, we just have to compose with continuous retractions to get functions defined on d^ω).

(f) $\Rightarrow$ (a) If $P \in$ pot$(\check{\mathbf{\Gamma}}_{c(\alpha)})$ separates A_0 from A_1 and (f) holds, then $S \subseteq (\Pi_{i \in d}\, f_i)^{-1}(P) \subseteq \neg(\lceil T_d \rceil \setminus S)$. This implies that S is separable from $\lceil T_d \rceil \setminus S$ by a pot$(\check{\mathbf{\Gamma}}_{c(\alpha)})$ set, by Lemma 4.4.7. But this contradicts the assumption on S. $\square$

Proof of Theorem 1.8.(1). Note first that (a) and (b) cannot hold simultaneously, as in the proof of Theorem 6.10.

We assume that (a) does not hold. This implies that the X_i's are not empty, since otherwise $A_0 = A_1 = \emptyset$, and $\emptyset \in \check{\mathbf{\Gamma}}$ unless $\mathbf{\Gamma} = \{\emptyset\}$. As in the proof of Theorem 4.1, we may assume that $X_i = \omega^\omega$ for each $i \in d$, by Lemma 4.4.7. By Theorem 5.1.3 there is $u \in \mathcal{D}$ with $\mathbf{\Gamma}(\omega^\omega) = \mathbf{\Gamma}_u(\omega^\omega)$. If E is a 0-dimensional Polish space, then we also have $\mathbf{\Gamma}(E) = \mathbf{\Gamma}_u(E)$, by Theorem 4.1.3 in [Lo-SR2]. It follows that pot$(\mathbf{\Gamma})$ = pot$(\mathbf{\Gamma}_u)$. By Lemmas 6.2 and 6.4 we may assume that there is $\alpha \in \Lambda^\infty$ normalized with $c(\alpha) = u$.

By Theorem 4.1.3 in [Lo-SR2] there is $B \in \mathbf{\Gamma}(\omega^\omega)$ with $S = j_d^{-1}(B) \cap \lceil T_d \rceil$. To simplify the notation, we may assume that T_d has $\varDelta^1_1$ levels, $\alpha \in \varDelta^1_1$, and $A_0, A_1 \in \varSigma^1_1\big((\omega^\omega)^d\big)$. By Lemma 6.5 there are $\beta, \gamma \in \omega^\omega$ such that $(\alpha, \beta, \gamma) \in \Upsilon^\infty_1$ and $C^{\omega^\omega}_\gamma = B$. Lemma 6.7.(b) gives $\underline{a}_0, \underline{a}_1, r$ with $(\alpha, a_0, a_1, \underline{a}_0, \underline{a}_1, r) \in \Theta^\infty$. Lemma 6.8 implies that $\neg\mathcal{U}_r \neq \emptyset$. So (b) holds, by Theorem 6.10. $\square$

The sequel is devoted to the proof of Theorem 6.9. We have to introduce a lot of objects before we can do it. We will create some paragraphs to describe these objects. We start with a general notion. The idea is that, given a set S in $\mathbf{\Gamma}_{c(\alpha)}(\lceil T_d \rceil)$, and with the help of the tree $\mathfrak{T}(\alpha)$, we will keep in mind all the $\mathbf{\Sigma}^0_\xi$ (or equivalently $\mathbf{\Pi}^0_\xi$, passing to complements) used to build S. We will represent these $\mathbf{\Pi}^0_\xi$ sets, on most sequences s of $\mathfrak{T}(\alpha)$, by induction on $|s|$, applying the Debs-Saint Raymond theorem. At each induction step, we make closed some $\mathbf{\Pi}^0_\xi$ sets of this level, but we also partially simplify the $\mathbf{\Pi}^0_\xi$ sets to come. This is why the ordinal substraction is involved (recall the definition of ordinal substraction after Theorem 5.1.3).

Definition 6.11 *Let X be a set, $A \subseteq X$, $\mathcal{B}$ a countable family of subsets of X, and $\mathbf{\Gamma}$ a Borel class. We say that $A \in \mathbf{\Gamma}(\mathcal{B})$ if $A \in \mathbf{\Gamma}(X, \tau)$ for any topology τ on X containing $\mathcal{B}$.*

Proposition 6.12 *Let X be a topological space.*

(a) Let $A \subseteq X$, $\mathcal{B}$ a countable family of open subsets of X, and $\mathbf{\Gamma}$ a Borel class. Then $A \in \mathbf{\Gamma}(X)$ if $A \in \mathbf{\Gamma}(\mathcal{B})$.

(b) Let Y be a set, $B \subseteq Y$, $f : X \to Y$ a bijection, $\mathcal{B}$ a countable family of subsets of Y, and $\mathbf{\Gamma}$ a Borel class. Then $f^{-1}(B) \in \mathbf{\Gamma}(\{f^{-1}(D) \mid D \in \mathcal{B}\})$ if $B \in \mathbf{\Gamma}(\mathcal{B})$.

(c) Let $1 \leq \eta \leq \xi$ and $A \in \mathbf{\Pi}^0_\xi(X)$. We assume that X is metrizable. Then there is $\mathcal{B} \subseteq \mathbf{\Pi}^0_\eta(X)$ countable such that $A \in \mathbf{\Pi}^0_{1+(\xi-\eta)}(\check{\mathcal{B}})$, where $\check{\mathcal{B}} := \{\neg B \mid B \in \mathcal{B}\}$.

In practice, X will be the metrizable space $[R]$ for some tree relation R, and f will be the canonical map given by the Debs-Saint Raymond theorem.

Proof. (a) The topology τ is simply the topology of X.

(b) Let τ be a topology on X containing $\{f^{-1}(D) \mid D \in \mathcal{B}\}$. Then $\sigma := \{f[A] \mid A \in \tau\}$ is a topology on Y containing $\mathcal{B}$. Thus $B \in \mathbf{\Gamma}(Y, \sigma)$ since $B \in \mathbf{\Gamma}(\mathcal{B})$. Therefore $f^{-1}(B) \in \mathbf{\Gamma}(X, \tau)$ since $f : (X, \tau) \to (Y, \sigma)$ is continuous.

(c) We argue by induction on $\xi-\eta$. The result is clear if $\xi-\eta=0$. So assume that $\xi-\eta\geq 1$. Write $A=\bigcap_{n\in\omega}\ \neg A_n$, where $\eta_n<\xi$ and $A_n\in\mathbf{\Pi}^0_{\eta_n}(X)$. As X is metrizable, we may assume that $\eta\leq\eta_n$. The induction assumption gives $\mathcal{B}_n\subseteq\mathbf{\Pi}^0_\eta(X)$ countable such that $A_n\in\mathbf{\Pi}^0_{1+(\eta_n-\eta)}(\check{\mathcal{B}}_n)$. It remains to set $\mathcal{B}:=\bigcup_{n\in\omega}\ \mathcal{B}_n$. $\square$

(A) The witnesses

Notation. We first define a map producing witnesses for the fact that $\vec{v}\in\Theta^\infty$. More specifically, we define a map $\mathfrak{W}:\Theta^\infty\rightarrow\Theta^\infty\cup(\Theta^\infty)^\omega$. Let $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\xi\setminus\Theta^{<\xi}$. If $|(\alpha)_1|=0$, then we set $\mathfrak{W}(\vec{v}):=\vec{v}$. If $|(\alpha)_1|=1$, then using the definition of Θ we set

$$\mathfrak{W}(\vec{v}):=(<(\alpha)_{2+j}>,a_0,a_1,\underline{a}_0,\underline{a}_1,a_1).$$

Note that $\mathfrak{W}(\vec{v})\in\Theta^{<\xi}$. If $|(\alpha)_1|=2$, then we set

$$\mathfrak{W}(\vec{v})(p):=\begin{cases}\big(<(\alpha)_{2+<0,q>}>,a_0,a_1,(a'_0)_0,(a'_1)_0,(r')_0\big)\text{ if }p=0,\\ \\ \big(<(\alpha)_{2+<(p)_0+1,q>}>,a_0,a_1,(a'_0)_p,(a'_1)_p,(r')_p\big)\text{ if }p\geq 1.\end{cases}$$

Here again, $\mathfrak{W}(\vec{v})(p)\in\Theta^{<\xi}$.

• Similarly, we define a map $\mathfrak{W}^1$ witnessing that $\vec{w}\in\Upsilon_1^\infty$. Moreover, we keep in mind γ'. More specifically, we define a map $\mathfrak{W}^1:\Upsilon_1^\infty\rightarrow\Upsilon_1^\infty\cup(\omega^\omega\times\Upsilon_1^\infty)\cup\big(\omega^\omega\times(\Upsilon_1^\infty)^\omega\big)$. Let $\vec{w}:=(\alpha,\beta,\gamma)$ in $\Upsilon_1^\xi\setminus\Upsilon_1^{<\xi}$. If $|(\alpha)_1|=0$, then we set $\mathfrak{W}^1(\vec{w}):=\vec{w}$. If $|(\alpha)_1|=1$, then using the definition of Υ_1 and some choice for γ', we set $\mathfrak{W}^1(\vec{w}):=\big(\gamma',(<(\alpha)_{2+j}>,\beta^*,\gamma')\big)$. If $|(\alpha)_1|=2$, then we set $\mathfrak{W}^1(\vec{w}):=\big(\gamma',\mathfrak{W}^1_1(\vec{w})\big)$, where

$$\mathfrak{W}^1_1(\vec{w})(p):=\begin{cases}\big(<(\alpha)_{2+<0,q>}>,(\beta^*)_0,(\gamma')_0\big)\text{ if }p=0,\\ \\ \big(<(\alpha)_{2+<(p)_0+1,q>}>,\big((\beta^*)_p\big)_0,\big((\gamma')_p\big)_0\big)\text{ if }p\geq 1.\end{cases}$$

(B) The trees associated with the codes for the non self-dual Wadge classes of Borel sets

• Recall the definition of $\mathfrak{T}(\alpha)$ after Lemma 6.2. Similarly, we define $\mathfrak{T}:\Upsilon_1^\infty\rightarrow\{\text{trees on }\omega\times\Upsilon_1^\infty\}$ as follows. Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\xi\setminus\Upsilon_1^{<\xi}$. We set

$$\mathfrak{T}(\vec{w}):=\begin{cases}\{\emptyset\}\cup\{<(0,\vec{w})>\}\text{ if }|(\alpha)_1|=0,\\ \{\emptyset\}\cup\big\{(0,\vec{w})^\frown s\mid s\in\mathfrak{T}\big(\mathfrak{W}^1_1(\vec{w})\big)\big\}\text{ if }|(\alpha)_1|=1,\\ \{\emptyset\}\cup\bigcup_{p\in\omega}\ \big\{(p,\vec{w})^\frown s\mid s\in\mathfrak{T}\big(\mathfrak{W}^1_1(\vec{w})(p)\big)\big\}\text{ if }|(\alpha)_1|=2.\end{cases}$$

Here again $\mathfrak{T}(\vec{w})$ is always a countable well founded tree containing the sequence $<(0,\vec{w})>$. The set of maximal sequences in $\mathfrak{T}(\vec{w})$ is $\mathcal{M}_{\vec{w}}:=\{s\in\mathfrak{T}(\vec{w})\mid\forall t\in\mathfrak{T}(\vec{w})\ \ s\subseteq t\Rightarrow s=t\}$.

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\Delta^1_1$ normalized. In the sequel, it will be convenient to set, for $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$,

$$s_1(|s|):=\begin{cases}\vec{w}\text{ if }s=\emptyset,\\ \mathfrak{W}^1_1\big(s_1(|s|-1)\big)\text{ if }s\neq\emptyset\wedge\big|\big(s_1(|s|-1)(0)\big)_1\big|=1,\\ \mathfrak{W}^1_1\big(s_1(|s|-1)\big)\big(s_0(|s|-1)\big)\text{ if }s\neq\emptyset\wedge\big|\big(s_1(|s|-1)(0)\big)_1\big|=2.\end{cases}$$

• Let $s\in\mathfrak{T}(\vec{w})$. We set $B_s:=\{i<|s| \mid |\big(s_1(i)(0)\big)_1|=2\}$. As α is normalized, B_s is an integer. We always have $B_s\leq |s|$. If moreover $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$, then we set $B'_s:=\{i\leq |s| \mid |\big(s_1(i)(0)\big)_1|=2\}$.

• The ordinals $|(\alpha)_0|$, for $\alpha\in\Delta^1_1\cap\Lambda^\infty$, will be of particular importance in the sequel. We define a function $\mathcal{Z}:\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}\to(\omega_1^{\mathrm{CK}})^{<\omega}$ satisfying $|\mathcal{Z}(s)|=|s|+1$. The sequence $\mathcal{Z}(s)$ gives the ordinals ξ of the $\mathbf{\Pi}^0_\xi$ sets coded by s. We set $\mathcal{Z}(s)(i):=\big|\big(s_1(i)(0)\big)_0\big|$ if $i\leq |s|$. Note the following properties of $\mathcal{Z}(s)$, easy to check:

- $\mathcal{Z}(s)(i)$ depends only on $s|i$.
- $\mathcal{Z}(s)\subseteq\mathcal{Z}(t)$ if $s\subseteq t$.
- $\mathcal{Z}(s)(i+1)\geq\mathcal{Z}(s)(i)$ or $\mathcal{Z}(s)(i+1)=0$ if $i<|s|$.
- $\mathcal{Z}(s)(i+1)=0$ if $\mathcal{Z}(s)(i)=0$ and $i<|s|$.
- $\big(\mathcal{Z}(s)(i)\big)_{i\in B'_s}$ is a non-decreasing sequence of non zero recursive ordinals.

(C) The resolution families

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\Delta^1_1$ normalized, and $p\geq 1$. We set

$$\tilde{P}^{\vec{w}}_p:=\begin{cases}\omega^\omega \text{ if } |(\alpha)_1|\leq 1,\\ C^{\omega^\omega}_{((\mathfrak{W}^1_0(\vec{w}))_p)_1} \text{ if } |(\alpha)_1|=2.\end{cases}$$

Note that $\tilde{P}^{\vec{w}}_p\in\mathbf{\Pi}^0_{|(\alpha)_0|}(\omega^\omega)$ if $|(\alpha)_1|=2$, by Lemma 6.1.

• Recall the finite sets $c_l\subseteq d^d$ defined at the end of the proof of Proposition 2.2 (we only used the fact that T_d has finite levels to see that they are finite). We put $c:=\bigcup_{l\in\omega}\ c_l$, so that c is countable. This will be the countable set c of Definition 4.3.1.

• Recall the embedding j_d defined before Theorem 6.9. We set $\mathcal{P}^{\vec{w}}_p:=h[j_d^{-1}(\tilde{P}^{\vec{w}}_p)\cap c^\omega]$, so that the union $\mathcal{P}^{\vec{w}}_p\cup\mathcal{P}^{\vec{w}}_q=[\subseteq]$ if $p\neq q\geq 1$. Moreover, $\mathcal{P}^{s_1(i)}_p\in\mathbf{\Pi}^0_{\mathcal{Z}(s)(i)}([\subseteq])$ if $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$ and $i\in B'_s$.

• If T is a tree and $s\in T$, then $T_s:=\{t\in T \mid s\subseteq t\}$.

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$. We say that $s\in\mathfrak{T}(\vec{w})$ is $extensible$ if there is $t\in\mathfrak{T}(\vec{w})_s$ such that $|s|<B_t$ (which implies that $s\notin\mathcal{M}_{\vec{w}}$). We will construct, for each s extensible, a resolution family $(R^{(\rho)}_s)_{\rho\leq\eta_s}$. Simultaneously, we construct some ordinals ξ_s and θ_s. If θ is an ordinal, then we set

$$\theta^*:=\begin{cases}\eta \text{ if } \theta=\eta+1,\\ \theta \text{ otherwise}\end{cases}$$

(this is what appears in the Debs-Saint Raymond theorem). We will have $\eta_s=\theta^*_s$, $\xi_s=\mathcal{Z}(s)(|s|)$ and

$$\theta_s:=\begin{cases}\xi_s=\mathcal{Z}(s)(0)=|(\alpha)_0| \text{ if } s=\emptyset,\\ 1+(\xi_s-\xi_{s^-}) \text{ if } s\neq\emptyset.\end{cases}$$

We want the resolution family to satisfy the following conditions:

- The family $(R_s^{(\rho)})_{\rho\leq\eta_s}$ is uniform if θ_s is a limit ordinal.
- $R_\emptyset^{(0)}={\subseteq}$, and $R_{s^-}^{(\eta_{s^-})}=R_s^{(0)}$ if $s\neq\emptyset$.
- $\Pi_s:[R_s^{(\eta_s)}]\to[R_s^{(0)}]$ is a continuous bijection.
- $(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_s)^{-1}(\mathcal{P}_p^{s_1(|s|)})\in\mathbf{\Pi}^0_1([R_s^{(\eta_s)}])$ if $p\geq 1$.
- $(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_s)^{-1}(\mathcal{P}_p^{t_1(j+1)})\in\mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\xi_s)}([R_s^{(\eta_s)}])$ if $p\geq 1$, $t\in\mathfrak{T}(\vec{w})_s\setminus\mathcal{M}_{\vec{w}}$ and $|s|<j+1\in B'_t$.

• The construction is by induction on $|s|$. Assume that $s=\emptyset$, $p\geq 1$, $t\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$ and $j+1\in B'_t$. Proposition 6.12.(c) gives $\mathcal{B}_p^{t,j}\subseteq\mathbf{\Pi}^0_{\theta_\emptyset}([\subseteq])$ countable such that $\mathcal{P}_p^{t_1(j+1)}\in\mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\theta_\emptyset)}(\check{\mathcal{B}}_p^{t,j})$. This implies that $u_\emptyset:=\{\mathcal{P}_p^{\vec{w}}\mid p\geq 1\}\cup\bigcup_{p\geq 1,t\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}},j+1\in B'_t}\ \mathcal{B}_p^{t,j}$ is countable and made of $\mathbf{\Pi}^0_{\theta_\emptyset}([\subseteq])$ sets. Theorems 4.3.4 and 4.4.4 give a family $(R_\emptyset^{(\rho)})_{\rho\leq\eta_\emptyset}$, uniform if $\theta_\emptyset$ is a limit ordinal, such that

- $R_\emptyset^{(0)}={\subseteq}$.
- $\Pi_\emptyset:[R_\emptyset^{(\eta_\emptyset)}]\to[R_\emptyset^{(0)}]$ is a continuous bijection.
- $\Pi_\emptyset^{-1}(Q)\in\mathbf{\Pi}^0_1([R_\emptyset^{(\eta_\emptyset)}])$ for each $Q\in u_\emptyset$.

This family is suitable, by Proposition 6.12.

• Assume now that $s\neq\emptyset$ is extensible, and the construction is done for the strict predecessors of s. Note that $(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_{s^-})^{-1}(\mathcal{P}_p^{s_1(|s|)})\in\mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{(\eta_{s^-})}])$. Assume that $p\geq 1$, $t\in\mathfrak{T}(\vec{w})_s\setminus\mathcal{M}_{\vec{w}}$ and $|s|<j+1\in B'_t$. Then Proposition 6.12.(c) gives a countable family $\mathcal{C}_p^{t,j}\subseteq\mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{\eta_{s^-}}])$ such that $(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_{s^-})^{-1}(\mathcal{P}_p^{t_1(j+1)})\in\mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\xi_s)}(\check{\mathcal{C}}_p^{t,j})$. This implies that

$$u_s:=\{(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_{s^-})^{-1}(\mathcal{P}_p^{s_1(|s|)})\mid p\geq 1\}\cup\bigcup_{p\geq 1,t\in\mathfrak{T}(\vec{w})_s\setminus\mathcal{M}_{\vec{w}},|s|<j+1\in B'_t}\mathcal{C}_p^{t,j}$$

is countable and made of $\mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{(\eta_{s^-})}])$ sets. Theorems 4.3.4 and 4.4.4 give a resolution family $(R_s^{(\rho)})_{\rho\leq\eta_s}$, uniform if θ_s is a limit ordinal, such that

- $R_s^{(0)}=R_{s^-}^{(\eta_{s^-})}$.
- $\Pi_s:[R_s^{(\eta_s)}]\to[R_s^{(0)}]$ is a continuous bijection.
- $\Pi_s^{-1}(Q)\in\mathbf{\Pi}^0_1([R_s^{(\eta_s)}])$ for each $Q\in u_s$.

This family is suitable, by Proposition 6.12. This completes the construction of the families.

(D) The subsets of T_d

We now build some subsets of T_d that will play the role that D and $T_d\setminus D$ played in the proof of Theorem 4.4.1. Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$. We will define a family of subsets of T_d as follows. Assume that $s\in\mathfrak{T}(\vec{w})$ is extensible. We set, for $q\geq 1$,

$$\begin{array}{ll}P_0(s):=\Big\{\vec{s}\in T_d\mid \vec{s}=\vec{\emptyset}\vee\forall p\geq 1\ \ \exists\mathcal{B}_p\in(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_s)^{-1}(\mathcal{P}_p^{s_1(|s|)})\ \ \vec{s}\in\mathcal{B}_p\Big\},\\ P_q(s):=\Big\{\vec{s}\in T_d\mid \vec{s}\neq\vec{\emptyset}\wedge\forall\mathcal{B}_q\in(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_s)^{-1}(\mathcal{P}_q^{s_1(|s|)})\ \ \vec{s}\notin\mathcal{B}_q\wedge\\ \qquad\qquad\forall p\in\omega\setminus\{0,q\}\ \ \exists\mathcal{B}_p\in(\Pi_{s|0}\circ\Pi_{s|1}\circ ...\circ\Pi_s)^{-1}(\mathcal{P}_p^{s_1(|s|)})\ \ \vec{s}\in\mathcal{B}_p\Big\}.\end{array}$$

Note that the $P_q(s)$'s are pairwise disjoint. The next lemma associates to each $\vec{t}\in T_d$ a sequence $s(\vec{t}\,)$ in $\mathfrak{T}(\vec{w})$ saying in which $P_q(s)$'s the sequence $\vec{t}$ is.

Proposition 6.13 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec{t}\in T_d$. Then there are $l\in\omega$ and $s(\vec{t}\,)\in\mathfrak{T}(\vec{w})$ of length l such that*
(a) $\vec{t}\in\bigcap_{i<l}\ P_{s(\vec{t}\,)(i)(0)}\big(s(\vec{t}\,)|i\big)$.
(b) If $s(\vec{t}\,)$ is extensible by t, then $\vec{t}\notin P_{t(l)(0)}(t|l)$.

Proof. We actually construct, for $j\in\omega$, a sequence $s_j\in\mathfrak{T}(\vec{w})$. We will have $s_j\subseteq s_{j+1}$, $|s_j|=j$ if $j\leq l$, $s_j=s_l$ if $j>l$, and $\vec{t}\in\bigcap_{i<|s_j|}\ P_{s_j(i)(0)}\big(s_j|i\big)$. At the end, $s(\vec{t}\,)$ will be s_l. The definition of s_j is by induction on j. Assume that $(s_k)_{k\leq j}$ are constructed satisfying these properties, which is the case for $j=0$. We may assume that $|s_j|=j$.

If s_j is not extensible or $\vec{t}\notin\mathcal{B}$ for each $\mathcal{B}\in[R_{s_j}^{(\eta_{s_j})}]$, then we set $s_{j+1}:=s_j$. If $\vec{t}\in\mathcal{B}$ for some $\mathcal{B}\in[R_{s_j}^{(\eta_{s_j})}]$, then there is a unique integer q such that $\vec{t}\in P_q(s_j)$ since

$$(\Pi_{s_j|0}\circ\Pi_{s_j|1}\circ ...\circ\Pi_{s_j})^{-1}(\mathcal{P}_p^{(s_j)_1(j)})\cup(\Pi_{s_j|0}\circ\Pi_{s_j|1}\circ ...\circ\Pi_{s_j})^{-1}(\mathcal{P}_q^{(s_j)_1(j)})=[R_{s_j}^{(\eta_{s_j})}]$$

if $p\neq q\geq 1$. We will have $|s_{j+1}|=j+1$, and $s_{j+1}(j)(0):=q$. Moreover,

$$s_{j+1}(j)(1):=\left\{\begin{array}{l}\vec{w}\text{ if }j=0,\\ \mathfrak{W}^1_1\big(s_j(j-1)(1)\big)\big(s_j(j-1)(0)\big)\text{ if }j\geq 1.\end{array}\right.$$

This completes the construction of the s_j's, and they are in $\mathfrak{T}(\vec{w})$. The well-foundedness of $\mathfrak{T}(\vec{w})$ proves the existence of l, and $s(\vec{t}\,)$ is suitable. □

Notation. Proposition 6.13 associates $s(\vec{t}\,)\in\mathfrak{T}(\vec{w})$ to $\vec{t}\in T_d$. Under the same conditions, we can associate $S(\vec{t}\,)\in\mathcal{M}_{\vec{w}}$ to $\vec{t}$. To do this, we need the following lemma:

Lemma 6.14 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $s\in\mathfrak{T}(\vec{w})$. Then there is $S\in\mathcal{M}_{\vec{w}}$ extending s such that $S_0(i)=0$ for $|s|\leq i<|S|$.*

Proof. If $s=\emptyset$, then we set $S(0):=(0,\vec{w})$ and, if $\mathfrak{W}^1\big(S_1(i)\big)\neq S_1(i)$, then we set

$$S(i+1):=\begin{cases}\Big(0,\mathfrak{W}^1_1\big(S(i)\big)\Big)\text{ if }\mathfrak{W}^1_1\big(S(i)\big)\in\Upsilon^\infty_1,\\ \\ \Big(0,\mathfrak{W}^1_1\big(S(i)\big)(0)\Big)\text{ if }\mathfrak{W}^1_1\big(S(i)\big)\in(\Upsilon^\infty_1)^\omega.\end{cases}$$

By induction, we see that $S|(i+1)\in\mathfrak{T}(\vec{w})$ for each $i<|S|$, which proves that the length of S is finite since $\mathfrak{T}(\vec{w})$ is well-founded. Thus $S\in\mathcal{M}_{\vec{w}}$.

If $s\neq\emptyset$, then $S(|s|-1)$ is defined. We argue similarly. The only thing to change is that

$$S(|s|):=\Big(0,\mathfrak{W}^1_1\big(s(|s|-1)\big)\big(s_0(|s|-1)\big)\Big)$$

if $\mathfrak{W}^1\big(s_1(|s|-1)\big)\neq s_1(|s|-1)$ and $\mathfrak{W}^1_1\big(s(|s|-1)\big)\in(\Upsilon^\infty_1)^\omega$. □

We now associate a maximal extension $S(\vec{t}\,)$ of $s(\vec{t}\,)$ to any $\vec{t}$ in T_d.

Remark. In particular, there is $S(\vec{\emptyset})\in\mathcal{M}_{\vec{w}}$ with $\big(S(\vec{\emptyset})\big)_0(i)=0$ for $i<|S(\vec{\emptyset})|$. Note that $s(\vec{\emptyset})\subseteq S(\vec{\emptyset})$. If $\vec{\emptyset}\neq\vec{t}\in T_d$, then we define $S(\vec{t}\,)$ by induction on $|\vec{t}|$:

- If $s(\vec{t}\,)=\emptyset$, then $\vec{t}\neq\emptyset$ since $\vec{\emptyset}\in P_0(\emptyset)$, and $S(\vec{t}\,):=S(\vec{t}^{\,\eta_\emptyset}_{\emptyset})$.
- If $s(\vec{t}\,)\neq\emptyset$ and $\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-}\in\bigcap_{i<|s(\vec{t}\,)|}P_{s(\vec{t}\,)(i)(0)}\big(s(\vec{t}\,)|i\big)$, then $S(\vec{t}\,):=S(\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-})$.
- If $s(\vec{t}\,)\neq\emptyset$ and $\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-}\notin\bigcap_{i<|s(\vec{t}\,)|}P_{s(\vec{t}\,)(i)(0)}\big(s(\vec{t}\,)|i\big)$, then $S(\vec{t}\,)$ is the extension of $s(\vec{t}\,)$ given by Lemma 6.14 applied to $s:=s(\vec{t}\,)$.

Note that $S(\vec{t}\,)\in\mathcal{M}_{\vec{w}}$ and is always an extension of $s(\vec{t}\,)$, by induction on $|\vec{t}|$. This comes from the fact that $s(\vec{t}\,)\subseteq s(\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-})$ in the second case.

(E) The tuples

We now keep in mind the tuples $(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)$ along any sequence of $\mathfrak{T}(\vec{w})$, using the witness map $\mathfrak{W}$. Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon^\infty_1$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$. We will define a map $V:\mathfrak{T}(\vec{w})\rightarrow(\Theta^\infty)^{<\omega}$ such that $|V(s)|=|s|$, $V(s)(i)$ depends only on $s|i$ as follows. We set, for $i<|s|$,

$$V(s)(i):=\begin{cases}\vec{v}\text{ if }i=0,\\ \mathfrak{W}\big(V(s)(i-1)\big)\text{ if }i\geq 1\wedge\big|\big(V(s)(i-1)(0)\big)_1\big|\leq 1,\\ \mathfrak{W}\big(V(s)(i-1)\big)\big(s_0(i-1)\big)\text{ if }i\geq 1\wedge\big|\big(V(s)(i-1)(0)\big)_1\big|=2.\end{cases}$$

Lemma 6.15 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon^\infty_1$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, $s\in\mathfrak{T}(\vec{w})$, and $i<|s|$. Then $V(s)(i)(0)=s_1(i)(0)$. In particular, $s\notin\mathcal{M}_{\vec{w}}$ and $i\leq|s|$ imply that $\mathcal{Z}(s)(i)=|\big(V(s)(i)(0)\big)_0|$.*

Proof. The last assertion clearly comes from the first one. The proof is by induction on i. The assertion is clear for $i=0$ since $V(s)(0)(0)=s_1(0)(0)=\alpha$. Assume that it holds for $i<|s|-1$.

• If $i\notin B_s$, then $|\big(V(s)(i)(0)\big)_1|=|\big(s_1(i)(0)\big)_1|=1$. Thus
$V(s)(i+1)(0)=\mathfrak{W}\big(V(s)(i)\big)(0)=<\big(V(s)(i)(0)\big)_{2+j}>=<\big(s_1(i)(0)\big)_{2+j}>=s_1(i+1)(0)$.

• If $i\in B_s$, then $\big|\big(V(s)(i)(0)\big)_1\big|=|\big(s_1(i)(0)\big)_1|=2$. If moreover $s_0(i)=0$, then

$$V(s)(i+1)(0)=<\big(V(s)(i)(0)\big)_{2+<0,q>}>=<\big(s_1(i)(0)\big)_{2+<0,q>}>=s_1(i+1)(0).$$

The argument is similar if $s_0(i)\geq 1$. □

The next lemma is a preparation for Lemma 6.21, which is the crucial step to prove a version of the claim in the proof of Theorem 4.4.1 for the non self-dual Wadge classes of Borel sets.

Lemma 6.16 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, $s\in\mathfrak{T}(\vec{w})$, and $i\in B_s$.*
(a) If $s_0(i)=0$, then $\neg\mathcal{U}_{V(s)(i)(5)}\subseteq\neg\mathcal{U}_{V(s)(i+1)(5)}$.
(b) We have $\neg\mathcal{U}_{V(s)(i)(5)}\subseteq\overline{\neg\mathcal{U}_{V(s)(i+1)(5)}}^{\tau_{\xi_{s|i}}}$.

Proof. (a) We have $V(s)(i+1)=\mathfrak{W}\big(V(s)(i)\big)(0)$, by Lemma 6.15. Thus

$$V(s)(i+1)(5)=\mathfrak{W}\big(V(s)(i)\big)(0)(5)=(r')_0$$

for some r' for which $\neg\mathcal{U}_{V(s)(i)(5)}\subseteq\neg\mathcal{U}_{(r')_0}$, by the 2nd and the 4th remarks after the definition of Θ.

(b) We may assume that $s_0(i)\geq 1$, so that $V(s)(i+1)(5)=(r')_{s_0(i)}$, and

$$\neg\mathcal{U}_{V(s)(i)(5)}\subseteq\overline{\neg\mathcal{U}_{V(s)(i+1)(5)}}^{\tau_{|(V(s)(i)(0))_0|}}$$

by the 5th remark after the definition of Θ and the definition of f_a. We are done by Lemma 6.15. □

(F) The sequences of integers

We have to keep in mind the integers $s_0(i)$ for $s\in\mathfrak{T}(\vec{w})$. We will consider an ordering of these finite sequences of integers that will help us to prove the claim just mentioned.

Notation. Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, and $s,s'\in\mathfrak{T}(\vec{w})$.

• If s and s' are not compatible, then we denote $s\wedge s':=s|i=s'|i$, where i is minimal with $s(i)\neq s'(i)$. Note that $|s\wedge s'|\in B_s$.

• We define $O(s)\in\omega^{|s|}$: we set $O(s)(i):=s_0(i)$.

• We also define a partial order on $\omega^{<\omega}$ as follows:

$$O\sqsubseteq O'\ \Leftrightarrow\ O=O'\vee\exists i<\min(|O|,|O'|)\ \big(O|i=O'|i\wedge O(i)=0<O'(i)\big).$$

Lemma 6.17 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, and $s,s'\in\mathfrak{T}(\vec{w})$ uncompatible. Assume that $\vec{s}\in\bigcap_{i\leq|s\wedge s'|}\ P_{s_0(i)}(s|i)$, $\vec{t}$ is in $\bigcap_{i\leq|s\wedge s'|}\ P_{s'_0(i)}(s'|i)$ and $\vec{s}\ R_{s||s\wedge s'|}^{(\eta_{s||s\wedge s'|})}\ \vec{t}$. Then $O(s)\sqsubseteq O(s')$.*

Proof. As $s(|s\wedge s'|)\neq s'(|s\wedge s'|)$ and $s_1(|s\wedge s'|)=s'_1(|s\wedge s'|)$, we get $s_0(|s\wedge s'|)\neq s'_0(|s\wedge s'|)$. Recall the definition of the $P_q(s)$'s. Note the following facts. Assume that $i\in B_s$ and $\vec{s}\ R_{s|i}^{(\eta_{s|i})}\ \vec{t}$.

- If $s_0(i)=0$ and $\vec{t}\in P_0(s|i)$, then $\vec{s}\in P_0(s|i)$ too.
- If $s_0(i)\geq 1$ and $\vec{t}\in P_{s_0(i)}(s|i)$, then $\vec{s}\in P_0(s|i)\cup P_{s_0(i)}(s|i)$.

These facts imply that $s_0(|s\wedge s'|)=0<s'_0(|s\wedge s'|)$. Therefore $O(s)\sqsubseteq O(s')$. $\square$

(G) The ranges

The goal of this paragraph is to defiine the analytic sets $r\big(S(\vec{t}\ )\big)$ that will contain $U_{\vec{t}}$ in the inductive construction of the proof of Theorem 6.9. They will play the role that $\overline{A_0}^{\tau_\xi}\cap A_1$ and A_0 played in the proof of Theorem 4.4.1, Conditions (4)-(5).

Notation. Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, and $s\in\mathfrak{T}(\vec{w})\setminus\{\emptyset\}$. We set

$$i^s:=\begin{cases}|s|-1 \text{ if } \forall j<|s|\ \ s_0(j)\geq 1,\\ \min\{i<|s|\mid s_0(i)=0\} \text{ otherwise,}\end{cases}$$

$$I^s:=\begin{cases}|s|-1 \text{ if } s_0(|s|-1)\geq 1,\\ \min\{i<|s|\mid \forall j\geq i\ \ s_0(j)=0\} \text{ otherwise.}\end{cases}$$

Note that $i^s\leq I^s\leq B_s$. We associate, with each $i^s\leq i<|s|$, $\underline{a}_0^{s,i},\underline{a}_1^{s,i},r^{s,i}\in\omega^\omega$. The definition is by induction on i. We set $\underline{a}_\varepsilon^{s,i^s}:=\underline{a}_\varepsilon\big(V(s)(i^s)(0),a_0,a_1\big)$, $r^{s,i^s}:=r\big(V(s)(i^s)(0),a_0,a_1\big)=V(s)(i^s)(5)$. Then

$$\underline{a}_\varepsilon^{s,i+1}:=\begin{cases}\underline{a}_\varepsilon^{s,i} \text{ if } s_0(i+1)\geq 1,\\ \underline{a}_\varepsilon\big(V(s)(i+1)(0),\underline{a}_0^{s,i},\underline{a}_1^{s,i}\big) \text{ if } s_0(i+1)=0,\end{cases}$$

$$r^{s,i+1}:=\begin{cases}r^{s,i} \text{ if } s_0(i+1)\geq 1,\\ r\big(V(s)(i+1)(0),\underline{a}_0^{s,i},\underline{a}_1^{s,i}\big) \text{ if } s_0(i+1)=0.\end{cases}$$

The *range* of s is $r(s):=\neg\mathcal{U}_{r^{s,I^s}}$.

Lemma 6.18 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, $s\in\mathfrak{T}(\vec{w})\setminus\{\emptyset\}$, and $i^s\leq i<B_s-1$ with $s_0(i)=0$. Then $r^{s,i}=r^{s,i+1}$.*

Proof. We may assume that $s_0(i+1)=0$. Assume first that $i=i^s$. Then

$$\begin{aligned}r^{s,i^s} &= r\big(V(s)(i^s)(0),a_0,a_1\big)\\ &= r\Big(\mathfrak{W}\big(V(s)(i^s)\big)(0)(0),\underline{a}_0\big(V(s)(i^s)(0),a_0,a_1\big),\underline{a}_1\big(V(s)(i^s)(0),a_0,a_1\big)\Big)\\ &= r\Big(\mathfrak{W}\big(V(s)(i^s)\big)\big(s_0(i^s)\big)(0),\underline{a}_0\big(V(s)(i^s)(0),a_0,a_1\big),\underline{a}_1\big(V(s)(i^s)(0),a_0,a_1\big)\Big)\\ &= r\Big(V(s)(i^s+1)(0),\underline{a}_0\big(V(s)(i^s)(0),a_0,a_1\big),\underline{a}_1\big(V(s)(i^s)(0),a_0,a_1\big)\Big)\\ &= r\big(V(s)(i^s+1)(0),\underline{a}_0^{s,i^s},\underline{a}_1^{s,i^s}\big)\\ &= r^{s,i^s+1}.\end{aligned}$$

The argument is similar if $i>i^s$. $\square$

Lemma 6.19 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$. Then there is $S(\vec{\emptyset})\in\mathcal{M}_{\vec{w}}$ with $\vec{\emptyset}\in\bigcap_{i<B_{S(\vec{\emptyset})}}P_{(S(\vec{\emptyset}))_0(i)}\big(S(\vec{\emptyset})|i\big)$ and $\neg\mathcal{U}_r\subseteq r\big(S(\vec{\emptyset})\big)$.*

Proof. We set $s:=S(\vec{\emptyset})$ for short. We already saw that $s\in\mathcal{M}_{\vec{w}}$, $\vec{\emptyset}\in\bigcap_{i<B_s}P_{s_0(i)}(s|i)$, and $s_0(i)=0$ for each $i<|s|$ after Lemma 6.14. Note that $i^s=I^s=0$. We get

$$\neg\mathcal{U}_r=\neg\mathcal{U}_{V(s)(0)(5)}=\neg\mathcal{U}_{V(s)(i^s)(5)}=\neg\mathcal{U}_{r^{s,i^s}}=\neg\mathcal{U}_{r^{s,I^s}}=r(s).$$

This finishes the proof. $\square$

The role of the next objects is to determine if we go to the A_0 side or the A_1 side in the inductive construction of the proof of Theorem 6.9.

Notation. Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, and $s\in\mathcal{M}_{\vec{w}}$. We set $\varepsilon_s:=0$ if $B_s<|s|-1$, $\varepsilon_s:=1$ otherwise, i.e., if $B_s=|s|-1$.

Lemma 6.20 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, and $s\in\mathcal{M}_{\vec{w}}$. Then $r(s)\subseteq\neg\mathcal{U}_{a_{\varepsilon_s}}$.*

Proof. Note first that $\neg\mathcal{U}_{\underline{a}_\varepsilon^{s,i}}\subseteq\neg\mathcal{U}_{a_\varepsilon}$, by induction on i and the 2nd remark after the definition of Θ. This implies that $\neg\mathcal{U}_{r^{s,I^s}}\subseteq\neg\mathcal{U}_{r(V(s)(I^s)(0),a_0,a_1)}=\neg\mathcal{U}_{V(s)(I^s)(5)}$, by the 4th remark after the definition of Θ. Thus $r(s)=\neg\mathcal{U}_{r^{s,I^s}}\subseteq\neg\mathcal{U}_{V(s)(I^s)(5)}$. Lemma 6.16 implies that $\neg\mathcal{U}_{V(s)(I^s)(5)}\subseteq\neg\mathcal{U}_{V(s)(B_s)(5)}$. But $V(s)(B_s)(5)=a_{\varepsilon_s}$, by Lemma 6.15. $\square$

Now we come to the crucial lemma for the claim mentioned earlier.

Lemma 6.21 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$, $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$ with $\alpha\in\varDelta_1^1$ normalized and $|(\alpha)_1|=2$, $s,s'\in\mathfrak{T}(\vec{w})$ with $O(s)\neq O(s')$ and $O(s)\sqsubseteq O(s')$. Then $r(s)\subseteq\overline{r(s')}^{\tau_{\xi_{s||s\wedge s'|}}}$.*

Proof. We can write $O(s):=0^{k_0}n_0...0^{k_{l-1}}n_{l-1}0^{k_l}$, with $l,k_i\in\omega$, and $n_i\geq 1$. Similarly, we write $O(s'):=0^{k'_0}n'_0...0^{k'_{l'-1}}n'_{l'-1}0^{k'_{l'}}$. The assumption implies that $l'\geq 1$, and the existence of $j<l'$ with $(k_i,n_i)=(k'_i,n'_i)$ if $i<j$ and $k'_j<k_j$. Lemma 6.14 shows the existence of $k''_{j+1}\geq 1$ and $s''\in\mathcal{M}_{\vec{w}}$ with $O(s'')=0^{k'_0}n'_0...0^{k'_{j-1}}n'_{j-1}0^{k'_j}n'_j0^{k''_{j+1}}$ if $j<l'-1$. If $j=l'-1$, then we set $s'':=s'$.

Note that $O(s) \neq O(s'')$, $O(s) \sqsubseteq O(s'')$, and $O(s'') \sqsubseteq O(s')$. Moreover, $O(s'') \neq O(s')$ and $|s \wedge s'| = |s \wedge s''| < |s' \wedge s''|$ if $j < l'-1$. It is enough to prove that $r(s) \subseteq \overline{r(s'')}^{\tau_{\xi_{s||s\wedge s''|}}}$. This means that we may assume that $(k_i, n_i) = (k'_i, n'_i)$ if $i < l'-1$ and $k'_{l'-1} < k_{l'-1}$. This implies that $I^{s'} \geq 1$, $|s \wedge s'| = I^{s'}-1$, $s|(I^{s'}-1) = s'|(I^{s'}-1)$, $s_0(I^{s'}-1) = 0 < s'_0(I^{s'}-1)$ and $i^s \leq I^{s'}-1$.

Case 1. $i^s = I^s$ and $i^{s'} = I^{s'}$.

Note that $r(s) = \neg\mathcal{U}_{r^{s,I^s}} = \neg\mathcal{U}_{r^{s,i^s}} = \neg\mathcal{U}_{V(s)(i^s)(5)} = \neg\mathcal{U}_{V(s')(I^s)(5)}$. Lemma 6.16 implies that

$$r(s) = \neg\mathcal{U}_{V(s')(I^s)(5)} \subseteq \neg\mathcal{U}_{V(s')(I^{s'}-1)(5)} \subseteq \overline{\neg\mathcal{U}_{V(s')(I^{s'})(5)}}^{\tau_{\xi_{s'|(I^{s'}-1)}}} = \overline{r(s')}^{\tau_{\xi_{s||s\wedge s'|}}}.$$

Case 2. $i^s = I^s$ and $i^{s'} < I^{s'}$.

Note that $i^s = i^{s'} < I^{s'}-1$. Lemma 6.18 implies that $r(s) = \neg\mathcal{U}_{r^{s,I^s}} = \neg\mathcal{U}_{r^{s,I^{s'}-1}}$. Thus

$$\begin{aligned}
r(s) &= \neg\mathcal{U}_{r(V(s)(I^{s'}-1)(0),\underline{a}_0^{s,I^{s'}-2},\underline{a}_1^{s,I^{s'}-2})} \\
&= \neg\mathcal{U}_{r(V(s')(I^{s'}-1)(0),\underline{a}_0^{s',I^{s'}-2},\underline{a}_1^{s',I^{s'}-2})} \\
&= \neg\mathcal{U}_{r(V(s')(I^{s'}-1)(0),\underline{a}_0^{s',I^{s'}-1},\underline{a}_1^{s',I^{s'}-1})} \\
&\subseteq \overline{\neg\mathcal{U}_{r(V(s')(I^{s'})(0),\underline{a}_0^{s',I^{s'}-1},\underline{a}_1^{s',I^{s'}-1})}}^{\tau_{\xi_{s'|(I^{s'}-1)}}} \\
&= \overline{r(s')}^{\tau_{\xi_{s||s\wedge s'|}}},
\end{aligned}$$

by Lemma 6.16.

Case 3. $i^s < I^s < I^{s'}$.

We argue as in Case 2.

Case 4. $i^s < I^s$ and $I^{s'} \leq I^s$, which implies that $I^{s'} < I^s$.

The 5th remark after the definition of Υ gives $\varepsilon \in 2$ with $r(s) = \neg\mathcal{U}_{r^{s,I^s}} \subseteq \neg\mathcal{U}_{\underline{a}_\varepsilon^{s,I^s-1}}$. Thus $r(s) \subseteq \neg\mathcal{U}_{\underline{a}_\varepsilon^{s,I^s-1}} \subseteq \ldots \subseteq \neg\mathcal{U}_{\underline{a}_\varepsilon^{s,I^{s'}-1}}$. If $I^{s'} \geq 2$, then we get

$$\begin{aligned}
\neg\mathcal{U}_{\underline{a}_\varepsilon^{s,I^{s'}-1}} &= \neg\mathcal{U}_{a_\varepsilon(V(s')(I^{s'}-1)(0),\underline{a}_0^{s',I^{s'}-2},\underline{a}_1^{s',I^{s'}-2})} \\
&\subseteq \overline{\neg\mathcal{U}_{r(V(s')(I^{s'})(0),\underline{a}_0^{s',I^{s'}-2},\underline{a}_1^{s',I^{s'}-2})}}^{\tau_{\xi_{s||s\wedge s'|}}} \\
&= \overline{\neg\mathcal{U}_{r(V(s')(I^{s'})(0),\underline{a}_0^{s',I^{s'}-1},\underline{a}_1^{s',I^{s'}-1})}}^{\tau_{\xi_{s||s\wedge s'|}}} \\
&= \overline{r(s')}^{\tau_{\xi_{s||s\wedge s'|}}}.
\end{aligned}$$

Otherwise, we get $I^{s'} = 1$, $i^s = 0$, $i^{s'} = I^{s'}$ and

$$\neg\mathcal{U}_{\underline{a}_\varepsilon^{s,0}} = \neg\mathcal{U}_{a_\varepsilon(V(s')(0)(0),a_0,a_1)} \subseteq \overline{\neg\mathcal{U}_{r(V(s')(1)(0),a_0,a_1)}}^{\tau_{\xi_{s||s\wedge s'|}}} = \overline{r(s')}^{\tau_{\xi_{s||s\wedge s'|}}}.$$

This finishes the proof. $\square$

(H) The maximal sequences

We now associate a maximal sequence to a couple $(\vec{\beta},\vec{w})$ with $\vec{\beta}\in\lceil T_d\rceil$. It is build in a way similar to that of the $s(\vec{t}\,)$'s, but for infinite sequences instead of finite ones.

• Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec{\beta}\in\lceil T_d\rceil$. We will define $s(\vec{\beta},\vec{w})\in\mathcal{M}_{\vec{w}}$. Recall the definition of $\tilde{\mathcal{P}}^{\vec{w}}_p$. We set, for $s\in\mathcal{M}_{\vec{w}}$ and $i\in B_s$,

$$E^s_i:=\begin{cases}\bigcap_{p\geq 1}\ \tilde{\mathcal{P}}^{s(i)(1)}_p \text{ if } s(i)(0)=0,\\ \neg\tilde{\mathcal{P}}^{s(i)(1)}_{s(i)(0)} \text{ if } s(i)(0)\geq 1.\end{cases}$$

We define $s(\vec{\beta},\vec{w})$ in such a way that $j_d(\vec{\beta})\in\bigcap_{i\in B_{s(\vec{\beta},\vec{w})}}\ E_i^{s(\vec{\beta},\vec{w})}$. Let ξ be an ordinal such that $\vec{w}\in\Upsilon_1^\xi\setminus\Upsilon_1^{<\xi}$. The definition of $s(\vec{\beta},\vec{w})$ is by induction on ξ.

Case 1. $|(\alpha)_1|=0$.

We set $s(\vec{\beta},\vec{w}):=<(0,\vec{w})>$.

Case 2. $|(\alpha)_1|=1$.

We set $s(\vec{\beta},\vec{w}):=(0,\vec{w})^\frown s\big(\vec{\beta},\mathfrak{W}^1_1(\vec{w})\big)$.

Case 3. $|(\alpha)_1|=2$.

We set $s(\vec{\beta},\vec{w}):=\begin{cases}(0,\vec{w})^\frown s\big(\vec{\beta},\mathfrak{W}^1_1(\vec{w})(0)\big) \text{ if } j_d(\vec{\beta})\in\bigcap_{p\geq 1}\ \tilde{\mathcal{P}}^{\vec{w}}_p,\\ (p,\vec{w})^\frown s\big(\vec{\beta},\mathfrak{W}^1_1(\vec{w})(p)\big) \text{ if } j_d(\vec{\beta})\notin\tilde{\mathcal{P}}^{\vec{w}}_p\wedge p\geq 1.\end{cases}$

• We set $(\vec{\beta}|j_k)_{k\in\omega}:=(\Pi_{s(\vec{\beta},\vec{w})|0}\circ ...\circ\Pi_{s(\vec{\beta},\vec{w})|(B_{s(\vec{\beta},\vec{w})}-1)})^{-1}\big(h(\vec{\beta})\big)$.

Recall the definition of ε_s before Lemma 6.20.

Lemma 6.22 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec{\beta}\in\lceil T_d\rceil$.*
(a) There is $k_0\in\omega$ such that $\vec{\beta}|j_k\in\bigcap_{i<B_{s(\vec{\beta},\vec{w})}}\ P_{s(\vec{\beta},\vec{w})(i)(0)}\big(s(\vec{\beta},\vec{w})|i\big)$ if $k\geq k_0$. In this case, the sequence $s(\vec{\beta}|j_k)$ given by Proposition 6.13 is $s(\vec{\beta},\vec{w})|B_{s(\vec{\beta},\vec{w})}$, and is not extensible.
(b) We have $j_d(\vec{\beta})\in C^{\omega^\omega}_\gamma$ if and only if $\varepsilon_{s(\vec{\beta},\vec{w})}=0$.

Proof. We set $s:=s(\vec{\beta},\vec{w})$ for simplicity.

(a) To define k_0, we will define, for $i<B_s$, $k^i_0\in\omega$ and we will set $k_0:=\max\{k^i_0\mid i<B_s\}$. To do this, we set $(\vec{\beta}|j^i_k)_{k\in\omega}:=(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}\big(h(\vec{\beta})\big)$, so that $(\vec{\beta}|j^{i+1}_k)_{k\in\omega}$ is a subsequence of $(\vec{\beta}|j^i_k)_{k\in\omega}$ if $i<B_s-1$.

By the choice of the E^s_i's we get, for $i<B_s$,

$$\begin{array}{ll}h(\vec\beta)\in & \begin{cases}\bigcap_{p\geq 1}\ \mathcal{P}_p^{s_1(i)} \text{ if } s_0(i)=0,\\ \neg\mathcal{P}_{s_0(i)}^{s_1(i)} \text{ if } s_0(i)\geq 1,\end{cases}\\ (\vec\beta|j^i_k)_{k\in\omega} & \in\begin{cases}\bigcap_{p\geq 1}\ (\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_p^{s_1(i)}) \text{ if } s_0(i)=0,\\ \neg(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)}) \text{ if } s_0(i)\geq 1.\end{cases}\end{array}$$

Note the existence of $\mathcal{B}^i_p$ in $(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_p^{s_1(i)})$ such that $\vec\beta|j^i_k\in\mathcal{B}^i_p$ if $s_0(i)=0$, $k\in\omega$ and $p\geq 1$. If $s_0(i)\geq 1$ and $p\in\omega\setminus\{0,s_0(i)\}$, then $(\vec\beta|j^i_k)_{k\in\omega}\in(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_p^{s_1(i)})$ since $\mathcal{P}_p^{s_1(i)}\cup\mathcal{P}_{s_0(i)}^{s_1(i)}=[\subseteq]$. This implies the existence of $\mathcal{B}^i_p\in(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_p^{s_1(i)})$ such that $\vec\beta|j^i_k\in\mathcal{B}^i_p$ if $k\in\omega$. As $(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)})\in\mathbf{\Pi}^0_1([R_{s|i}^{(\eta_{s|i})}])$, there is $k^i_0\geq 1$ such that $\vec\beta|j^i_k\notin\mathcal{B}^i_{s_0(i)}$ if $s_0(i)\geq 1$, $\mathcal{B}^i_{s_0(i)}\in(\Pi_{s|0}\circ ...\circ\Pi_{s|i})^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)})$ and $k\geq k^i_0$. This defines k^i_0 and k_0. It remains to check that $\vec\beta|j_k\in P_{s(i)(0)}(s|i)$ if $i<B_s$ and $k\geq k_0$. This comes from the fact that $j_k=j_k^{B_s-1}=j^i_{K(k)}$ for some $K(k)\geq k\geq k_0\geq k^i_0$. The last assertion comes from the construction of $s(\vec t\,)$.

(b) We define, for $i<|s|$, $\varepsilon^i_s\in 2$. The definition is by induction on i. We first set $\varepsilon^0_s:=1$. Then $\varepsilon^{i+1}_s:=0$ if $|s|-i-2\notin B_s$, $\varepsilon^{i+1}_s:=\varepsilon^i_s$ otherwise. Note that $\varepsilon_s=\varepsilon_s^{|s|-1}$ (ε_s is defined before Lemma 6.20). We have to see that $j_d(\vec\beta)$ is in $C^{\omega^\omega}_{s_1(0)(2)}$ is equivalent to $\varepsilon_s^{|s|-1}=0$. We prove the following stronger fact: $j_d(\vec\beta)\in C^{\omega^\omega}_{s_1(|s|-i-1)(2)}$ is equivalent to $\varepsilon^i_s=0$ if $i<|s|$. Here again we argue by induction on i. The result is clear for $i=0$ since $C^{\omega^\omega}_{s_1(|s|-1)(2)}=\emptyset$. So assume that the result is true for $i<|s|-1$.

If $|s|-i-2\notin B_s$, then we are done since $\varepsilon^{i+1}_s=1-\varepsilon^i_s$ and $C^{\omega^\omega}_{s_1(|s|-i-2)(2)}=\neg C^{\omega^\omega}_{s_1(|s|-i-1)(2)}$. If $|s|-i-2\in B_s$, then $\varepsilon^{i+1}_s=\varepsilon^i_s$ and

$$\begin{array}{ll}C^{\omega^\omega}_{s_1(|s|-i-2)(2)}=\bigcup_{p\geq 1}\ (C^{\omega^\omega}_{((\mathfrak{W}^1_0(s_1(|s|-i-2)))_p)_0}\setminus C^{\omega^\omega}_{((\mathfrak{W}^1_0(s_1(|s|-i-2)))_p)_1})\cup\\ \qquad\qquad (C^{\omega^\omega}_{(\mathfrak{W}^1_0(s_1(|s|-i-2)))_0}\cap\bigcap_{p\geq 1}\ C^{\omega^\omega}_{((\mathfrak{W}^1_0(s_1(|s|-i-2)))_p)_1}).\end{array}$$

If $s_0(|s|-i-2)=0$, then $j_d(\vec\beta)\in\bigcap_{p\geq 1}\ \tilde{\mathcal{P}}_p^{s_1(|s|-i-2)}=\bigcap_{p\geq 1}\ C^{\omega^\omega}_{((\mathfrak{W}^1_0(s_1(|s|-i-2)))_p)_1}$. We can say that $j_d(\vec\beta)\in C^{\omega^\omega}_{s_1(|s|-i-2)(2)}$ is equivalent to $j_d(\vec\beta)\in C^{\omega^\omega}_{(\mathfrak{W}^1_0(s_1(|s|-i-2)))_0}=C^{\omega^\omega}_{s_1(|s|-i-1)(2)}$, and we are done by induction assumption. We argue similarly when $s_0(|s|-i-2)\geq 1$. $\square$

Remark. Recall the definition of an extensible sequence at the beginning of the construction of the resolution families. If s is not extensible, then s admits a unique extension $M(s)$ in $\mathcal{M}_{\vec w}$. In particular, in Lemma 6.22.(a), $M\big(s(\vec\beta|j_k)\big)=s(\vec\beta,\vec w)=S(\vec\beta|j_k)$. In Lemma 6.19, $s(\vec\emptyset)=s|B_s$ is not extensible and $M\big(s(\vec\emptyset)\big)=S(\vec\emptyset)$.

Notation. Recall the construction of the resolution families, and also the proof of Theorem 4.4.5, especially the definition of $\eta(\vec t\,)$. If θ_s is a limit ordinal, then we consider some ordinals $\eta_s(\vec t\,)$'s, as in the proof of Theorem 4.4.5. We set $\rho(s,\vec s\,):=\begin{cases}\eta_s \text{ if } \theta_s \text{ is a successor ordinal,}\\ \eta_s(\vec s\,) \text{ if } \theta_s \text{ is a limit ordinal.}\end{cases}$

The next lemma is the final preparation for the claim mentioned earlier.

Lemma 6.23 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, $s\in\mathfrak{T}(\vec{w})$, and $i<B_s$. Then $\big(\Sigma_{i'\leq i}\ \rho(s|i',\overrightarrow{tm})\big)+1\leq\xi_{s|i}$.*

Proof. We argue by induction on i. Note first that $\rho(s|0,\overrightarrow{tm})+1\leq\theta_{s|0}=\xi_{s|0}$. Then, inductively,

$$\begin{aligned}\big(\Sigma_{i'\leq i+1}\ \rho(s|i',\overrightarrow{tm})\big)+1 &\leq \big(\Sigma_{i'\leq i}\ \rho(s|i',\overrightarrow{tm})\big)+\theta_{s|(i+1)}\\ &\leq \big(\Sigma_{i'\leq i}\ \rho(s|i',\overrightarrow{tm})\big)+1+(\xi_{s|(i+1)}-\xi_{s|i})\\ &\leq \xi_{s|i}+(\xi_{s|(i+1)}-\xi_{s|i})\\ &\leq \xi_{s|(i+1)}\end{aligned}$$

This finishes the proof. $\square$

Proof of Theorem 6.9. Let ξ be an ordinal with $\vec{w}:=(\alpha,\beta,\gamma)\in\Upsilon_1^\xi$. We argue by induction on ξ. So assume that $\vec{w}\in\Upsilon_1^\xi\setminus\Upsilon_1^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Lemma 6.5 implies that $C_\gamma^{\omega^\omega}\in\mathbf{\Gamma}_{c(\alpha)}=\mathbf{\Gamma}_{0^\infty}=\{\emptyset\}$, so that $S=\emptyset$. We also have $r=a_1$. Assume that (a) does not hold. Then $A_1\neq\emptyset$, so it contains some $\vec{\alpha}$. We just have to set $f_i(\beta_i):=\alpha_i$.

Case 2. $|(\alpha)_1|=1$.

As $\vec{w}\in\Upsilon_1^\xi$ we get $\gamma'\in\omega^\omega$ with $(<(\alpha)_{2+j}>,\beta^*,\gamma')\in\Upsilon_1^{<\xi}$ and $C_\gamma^{\omega^\omega}=\neg C_{\gamma'}^{\omega^\omega}$ (see the definition of Υ_1). As α is normalized, $C_{\gamma'}^{\omega^\omega}=\emptyset$, so that $S=\lceil T_d\rceil$. We also have $r=a_0$. Assume that (a) does not hold. Then $A_0\neq\emptyset$, and we argue as in Case 1.

Case 3. $|(\alpha)_1|=2$.

Assume that (a) does not hold. As for Theorems 4.4.1 and 4.4.5 we construct $(\alpha_s^i)_{i\in d,s\in\Pi_i''T_d}$, $(O_s^i)_{i\leq|s|,i\in d,s\in\Pi_i''T_d}$, $(U_{\vec{s}})_{\vec{s}\in T_d}$. We want these objects to satisfy the following conditions.

$$(1)\ \alpha_s^i\in O_s^i\subseteq\Omega_{\omega^\omega}\wedge(\alpha_{s_i}^i)_{i\in d}\in U_{\vec{s}}\subseteq\Omega_{(\omega^\omega)^d},$$

$$(2)\ O_{sq}^i\subseteq O_s^i,$$

$$(3)\ \text{diam}_{d_{\omega^\omega}}(O_s^i)\leq 2^{-|s|}\wedge\text{diam}_{d_{(\omega^\omega)^d}}(U_{\vec{s}})\leq 2^{-|\vec{s}|},$$

$$(4)\ \vec{t}\in T_d\ \Rightarrow\ U_{\vec{t}}\subseteq r\big(S(\vec{t}\,)\big),$$

$$(5)\ \left(\begin{array}{c}\vec{s},\vec{t}\in\bigcap_{i'<i,\eta_{s|i'}\geq 1}\ P_{s_0(i')}(s|i')\\ 1\leq\rho\leq\rho(s|i,\vec{s})\\ \vec{s}\ R_{s|i}^{(\rho)}\ \vec{t}\end{array}\right)\ \Rightarrow\ U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^{\tau(\Sigma_{i'<i}\ \rho(s|i',\vec{s}))+\rho},$$

$$(6)\ \Big(\vec{s}\in\bigcap_{i<|s(\vec{t}\,)|}\ P_{s(\vec{t}\,)(i)(0)}\big(s(\vec{t}\,)|i\big)\wedge\vec{s}\ R_{s(\vec{t}\,)^-}^{(\eta_{s(\vec{t}\,)^-})}\ \vec{t}\Big)\ \Rightarrow\ U_{\vec{t}}\subseteq U_{\vec{s}}.$$

• Let us prove that this construction is sufficient to get the theorem.

- Fix $\vec{\beta}\in\lceil T_d\rceil$. Lemma 6.22 gives $k_0\in\omega$ such that $\vec{\beta}|j_k\in\bigcap_{i<B_{s(\vec{\beta},\vec{w})}}\ P_{s(\vec{\beta},\vec{w})(i)(0)}\big(s(\vec{\beta},\vec{w})|i\big)$ for each $k\geq k_0$. Proposition 6.13 gives l and $s(\vec{\beta}|j_k)\in\mathfrak{T}(\vec{w})$ with $\vec{\beta}|j_k\in\bigcap_{i<l}\ P_{s(\vec{\beta}|j_k)(i)(0)}\big(s(\vec{\beta}|j_k)|i\big)$, and Lemma 6.22.(a) implies that $s(\vec{\beta}|j_k)=s(\vec{\beta},\vec{w})|B_{s(\vec{\beta},\vec{w})}$. This implies that $(U_{\vec{\beta}|j_k})_{k\geq k_0}$ is non-increasing since $\vec{\beta}|j_k\ R_{s(\beta,\vec{w})|(B_{s(\beta,\vec{w})}-1)}^{(\eta_{s(\beta,\vec{w})|(B_{s(\beta,\vec{w})}-1)})}\ \vec{\beta}|j_{k+1}$ for each integer k, by Condition (6). As in the proof of Theorem 4.4.1 we define $\mathcal{F}(\vec{\beta})$ and f_i continuous with $\mathcal{F}(\vec{\beta})=(\Pi_{i\in d}\ f_i)(\vec{\beta})$. The inclusions

$$S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$$

and $\lceil T_d\rceil\setminus S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$ hold, by Lemmas 6.20 and 6.22, since $r\big(s(\beta,\vec{w})\big)\subseteq A_{\varepsilon_{s(\beta,\vec{w})}}$.

• So let us prove that the construction is possible.

- As $\neg\mathcal{U}_r$ is nonempty and Σ^1_1, we can choose $(\alpha^i_\emptyset)_{i\in d}\in\neg\mathcal{U}_r\cap\Omega_{(\omega^\omega)^d}$. Then we choose a Σ^1_1 subset $U_{\vec{\emptyset}}$ of $(\omega^\omega)^d$, with $d_{(\omega^\omega)^d}$-diameter at most 1, such that $(\alpha^i_\emptyset)_{i\in d}\in U_{\vec{\emptyset}}\subseteq\neg\mathcal{U}_r\cap\Omega_{(\omega^\omega)^d}$. We choose a Σ^1_1 subset $O^0_\emptyset$ of Ω_{ω^ω}, with d_{ω^ω}-diameter at most 1, with $\alpha^0_\emptyset\in O^0_\emptyset\subseteq\Omega_{\omega^\omega}$, which is possible since $\Omega_{(\omega^\omega)^d}\subseteq\Omega^d_{\omega^\omega}$. Assume that $(\alpha^i_s)_{|s|\leq l}$, $(O^i_s)_{|s|\leq l}$ and $(U_{\vec{s}})_{|s_0|\leq l}$ satisfying conditions (1)-(6) have been constructed, which is the case for $l=0$ by Lemma 6.19.

- Let $\overrightarrow{tm}\in T_d\cap(d^{l+1})^d$. We define $X_i:=O^i_{t_i}$ if $i\leq l$, and ω^ω if $i>l$.

Claim. *Assume that $s\in\mathfrak{T}(\vec{w})$, $i<B_s$, $\overrightarrow{tm}^{\eta_{s|i}}_{s|i},\overrightarrow{tm}\in\bigcap_{i'<i}\ P_{s_0(i')}(s|i')$, and $i_0\leq i$ is minimal with $\eta_{s|i_0}\geq 1$.*
(a) The set

$$U_{\overrightarrow{tm}^{\rho(s|i,\overrightarrow{tm})}_{s|i}}\cap\bigcap_{1\leq\rho<\rho(s|i,\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s|i}}}^{\tau_{(\Sigma_{i'<i}\ \rho(s|i',\overrightarrow{tm}))+\rho}}$$
$$\cap\bigcap_{i'<i}\ \bigcap_{1\leq\rho\leq\rho(s|i',\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s|i'}}}^{\tau_{(\Sigma_{i''<i'}\ \rho(s|i'',\overrightarrow{tm}))+\rho}}\cap(\Pi_{i\in d}\ X_i)$$

is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1_{s|i_0}}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$.

(b) Assume moreover that $s'\in\mathfrak{T}(\vec{w})$, s and s' are uncompatible, $i:=|s\wedge s'|$, $\overrightarrow{tm}\in P_{s'_0(i)}(s'|i)$, and $\overrightarrow{tm}^{\eta_{s|i}}_{s|i}\in P_{s_0(i)}(s|i)$. Then

$$r\big(S(\overrightarrow{tm})\big)\cap\bigcap_{i'\leq i}\ \bigcap_{1\leq\rho\leq\rho(s|i',\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s|i'}}}^{\tau_{(\Sigma_{i''<i'}\ \rho(s|i'',\overrightarrow{tm}))+\rho}}\cap(\Pi_{i\in d}\ X_i)$$

is τ_1-dense in $\overline{U_{\overrightarrow{tm}^1_{s|i_0}}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$.

(a) Assume first that $i_0=0$. Note that $\overrightarrow{tm}^{\rho+1}_\emptyset\ R^{(\rho+1)}_\emptyset\ \overrightarrow{tm}^{\rho}_\emptyset\ R^{(\rho)}_\emptyset\ \overrightarrow{tm}$ if $1\leq\rho<\rho(\emptyset,\overrightarrow{tm})$, by Lemma 4.3.2. As in the proof of Claim 2 in Theorem 4.4.5, this implies that $U_{\overrightarrow{tm}^{\rho}_\emptyset}\subseteq\overline{U_{\overrightarrow{tm}^{\rho+1}_\emptyset}}^{\tau_{\rho+1}}$. By assumption, $\overrightarrow{tm}^{\eta_{s|i}}_{s|i},\overrightarrow{tm}\in\bigcap_{i'<i}\ P_{s_0(i')}(s|i')$.

Note that $\overrightarrow{tm}^{\rho}_{s|(i'+1)} \in P_{s_0(i'')}(s|i'')$ if $i'' \leq i' < i$ and $\rho \leq \eta_{s|(i'+1)}$. Indeed, this comes from the fact that $\overrightarrow{tm}^{\eta_{s|i}}_{s|i}\ R^{(\eta_{s|i''})}_{s|i''}\ \overrightarrow{tm}^{\rho}_{s|(i'+1)}\ R^{(\eta_{s|i''})}_{s|i''}\ \overrightarrow{tm}$. As in the proof of Claim 2 in Theorem 4.4.5 again, this implies that $U_{\overrightarrow{tm}^{\rho}_{s|(i'+1)}} \subseteq \overline{U_{\overrightarrow{tm}^{\rho+1}_{s|(i'+1)}}}^{\tau_{(\Sigma_{i''<i'+1}\ \rho(s|i'',\overrightarrow{tm}))+\rho+1}}$ if $\rho < \rho(s|(i'+1),\overrightarrow{tm})$. Note that $\overrightarrow{tm}^{0}_{s|(i'+1)} = \overrightarrow{tm}^{\eta_{s|i'}}_{s|i'} = \overrightarrow{tm}^{\rho(s|i',\overrightarrow{tm})}_{s|i'}$. This implies the result. We argue similarly if $i_0 > 0$.

(b) By (a) and Lemma 6.22, it is enough to see that $U := U_{\overrightarrow{tm}^{\rho(s|i,\overrightarrow{tm})}_{s|i}} \subseteq \overline{r\big(S(\overrightarrow{tm})\big)}^{\tau_{\xi_{s|i}}}$. The induction assumption implies that $U \subseteq r\big(S(\overrightarrow{tm}^{\eta_{s|i}}_{s|i})\big)$. So let us prove that $r\big(S(\overrightarrow{tm}^{\eta_{s|i}}_{s|i})\big) \subseteq \overline{r\big(S(\overrightarrow{tm})\big)}^{\tau_{\xi_{s|i}}}$. Note that $s|(i+1) \subseteq s(\overrightarrow{tm}^{\eta_{s|i}}_{s|i}) \subseteq S(\overrightarrow{tm}^{\eta_{s|i}}_{s|i})$ and, similarly, $s'|(i+1) \subseteq S(\overrightarrow{tm})$. Lemma 6.17 implies that $O\big(S(\overrightarrow{tm}^{\eta_{s|i}}_{s|i})\big) \sqsubseteq O\big(S(\overrightarrow{tm})\big)$, and the beginning of its proof that $O\big(S(\overrightarrow{tm}^{\eta_{s|i}}_{s|i})\big) \neq O\big(S(\overrightarrow{tm})\big)$. Now Lemma 6.21 implies the result. $\diamond$

- Let $\mathcal{X} := d^{l+1}$. The map $\Psi : \mathcal{X}^d \to \varSigma^1_1\big((\omega^\omega)^d\big)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Psi(\overrightarrow{tm}) := \left\{ \begin{array}{l} r\big(S(\overrightarrow{tm})\big) \cap \bigcap_{1\leq\rho\leq\rho(\emptyset,\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{\emptyset}}}^{\tau_\rho} \cap (\Pi_{i\in d}\ X_i) \cap \Omega_{(\omega^\omega)^d} \text{ if } s(\overrightarrow{tm}) = \emptyset, \\ \\ U_{\overrightarrow{tm}^{\rho(s(\overrightarrow{tm})^-,\overrightarrow{tm})}_{s(\overrightarrow{tm})^-}} \cap \bigcap_{1\leq\rho<\rho(s(\overrightarrow{tm})^-,\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s(\overrightarrow{tm})^-}}}^{\tau_{(\Sigma_{i'<|s(\overrightarrow{tm})|-1}\ \rho(s|i',\overrightarrow{tm}))+\rho}} \\ \hfill \cap \bigcap_{i'<|s(\overrightarrow{tm})|-1}\ \bigcap_{1\leq\rho\leq\rho(s|i',\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s|i'}}}^{\tau_{(\Sigma_{i''<i'}\ \rho(s|i'',\overrightarrow{tm}))+\rho}} \cap (\Pi_{i\in d}\ X_i) \\ \hfill \text{if } s(\overrightarrow{tm}) \neq \emptyset \wedge \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})^-}}_{s(\overrightarrow{tm})^-} \in \bigcap_{i'<|s(\overrightarrow{tm})|}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge \exists i_0 < |s(\overrightarrow{tm})|\ \ \eta_{s(\overrightarrow{tm})|i_0} \geq 1, \\ \\ r\big(S(\overrightarrow{tm})\big) \\ \cap \bigcap_{i'\leq i}\ \bigcap_{1\leq\rho\leq\rho(s(\overrightarrow{tm})|i',\overrightarrow{tm})}\ \overline{U_{\overrightarrow{tm}^{\rho}_{s(\overrightarrow{tm})|i'}}}^{\tau_{(\Sigma_{i''<i'}\ \rho(s(\overrightarrow{tm})|i'',\overrightarrow{tm}))+\rho}} \cap (\Pi_{i\in d}\ X_i) \cap \Omega_{(\omega^\omega)^d} \\ \hfill \text{if } s(\overrightarrow{tm}) \neq \emptyset \wedge \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})^-}}_{s(\overrightarrow{tm})^-} \notin \bigcap_{i'<|s(\overrightarrow{tm})|}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge\ i < |s(\overrightarrow{tm})| \text{ is maximal with } \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})|i}}_{s(\overrightarrow{tm})|i} \in \bigcap_{i'<i}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge\ \exists i_0 \leq i\ \ \eta_{s(\overrightarrow{tm})|i_0} \geq 1, \\ \\ U_{\vec{t}} \cap (\Pi_{i\in d}\ X_i) \text{ if } s(\overrightarrow{tm}) \neq \emptyset \wedge \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})^-}}_{s(\overrightarrow{tm})^-} \in \bigcap_{i'<|s(\overrightarrow{tm})|}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge\ \forall i_0 < |s(\overrightarrow{tm})|\ \ \eta_{s(\overrightarrow{tm})|i_0} = 0, \\ \\ r\big(S(\overrightarrow{tm})\big) \cap (\Pi_{i\in d}\ X_i) \cap \Omega_{(\omega^\omega)^d} \ \text{ if } \ s(\overrightarrow{tm}) \neq \emptyset \\ \hfill \wedge\ \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})^-}}_{s(\overrightarrow{tm})^-} \notin \bigcap_{i'<|s(\overrightarrow{tm})|}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge\ i < |s(\overrightarrow{tm})| \text{ is maximal with } \overrightarrow{tm}^{\eta_{s(\overrightarrow{tm})|i}}_{s(\overrightarrow{tm})|i} \in \bigcap_{i'<i}\ P_{s(\overrightarrow{tm})(i')(0)}\big(s(\overrightarrow{tm})|i'\big) \\ \hfill \wedge\ \forall i_0 \leq i\ \ \eta_{s(\overrightarrow{tm})|i_0} = 0. \end{array} \right.$$

By the claim, $\Psi(\overrightarrow{tm})$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}^{1}_{s(\overrightarrow{tm})|i_0}}}^{\tau_1} \cap (\Pi_{i\in d}\ X_i)$ in the 2nd and in the 3rd cases.

In these cases, as $\overrightarrow{tm}^1_{s(\overrightarrow{tm})|i_0}\subseteq\vec{t}\subseteq\overrightarrow{tm}$ and $R^{(1)}_{s(\overrightarrow{tm})|i_0}$ is distinguished in $R^{(0)}_{s(\overrightarrow{tm})|i_0}=\subseteq$,

$$\overrightarrow{tm}^1_{s(\overrightarrow{tm})|i_0}\ R^{(1)}_{s(\overrightarrow{tm})|i_0}\ \vec{t}$$

and $U_{\vec{t}}\subseteq\overline{U_{\overrightarrow{tm}^1_{s(\overrightarrow{tm})|i_0}}}^{\tau_1}$, by induction assumption. Therefore

$$U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{U_{\overrightarrow{tm}^1_{s(\overrightarrow{tm})|i_0}}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Psi}(\overrightarrow{tm}).$$

Using similar arguments, one can prove that this also holds in the last two cases.

Let us look at the first case. If $\eta_\emptyset\geq 1$, then using arguing as in the claim one can prove that $U_{\overrightarrow{tm}_\emptyset^{\rho(\emptyset,\overrightarrow{tm})}}\cap\bigcap_{1\leq\rho<\rho(\emptyset,\overrightarrow{tm})}\overline{U_{\overrightarrow{tm}_\emptyset^\rho}}^{\tau_\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{\overrightarrow{tm}_\emptyset^1}}^{\tau_1}\cap(\Pi_{i\in d}\ X_i)$. Now we can write $U_{\overrightarrow{tm}_\emptyset^{\rho(\emptyset,\overrightarrow{tm})}}\subseteq r\big(S(\overrightarrow{tm}_\emptyset^{\eta_\emptyset})\big)=r\big(S(\overrightarrow{tm})\big)$ and we can repeat the previous argument since $i_0=0$. If $\eta_\emptyset=0$, then we get $\overrightarrow{tm}_\emptyset^{\eta_\emptyset}=\vec{t}$, and $U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq r\big(S(\vec{t}\,)\big)\cap(\Pi_{i\in d}\ X_i)=r\big(S(\overrightarrow{tm})\big)\cap(\Pi_{i\in d}\ X_i)$ and we are done.

Now we can write $(\alpha^i_{t_i})_{i\in d}\in U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Psi}(\overrightarrow{tm})$, and we conclude as in the proof of Theorem 4.4.1. $\square$

The rest of this section is devoted to the proof of Theorem 1.8.(2) when $\Delta(\mathbf{\Gamma})$ is a Wadge class. Recall Theorem 5.2.8. We will say that $\alpha\in\Delta^1_1\cap\Lambda^\infty$ is *suitable* if $\Delta(\mathbf{\Gamma}_{c(\alpha)})$ is a Wadge class and one of the following holds:

(1) There is $\overline{\alpha}\in\Delta^1_1\cap\Lambda^\infty$ normalized with

$$\mathbf{\Gamma}_{c(\alpha)}=\Big\{(A_0\cap C_0)\cup(A_1\cap C_1)\mid A_0,\neg A_1\in\mathbf{\Gamma}_{c(\overline{\alpha})}\wedge C_0,C_1\in\mathbf{\Sigma}^0_1\wedge C_0\cap C_1=\emptyset\Big\}.$$

(2) There is $\alpha'\in\Delta^1_1$ such that $(\alpha')_p\in\Lambda^\infty$ is normalized for each $p\geq 1$, $\big(\mathbf{\Gamma}_{c((\alpha')_p)}\big)_{p\geq 1}$ is strictly increasing, and $\mathbf{\Gamma}_{c(\alpha)}=\Big\{\bigcup_{p\geq 1}\ (A_p\cap C_p)\mid A_p\in\mathbf{\Gamma}_{c((\alpha')_p)}\wedge C_p\in\mathbf{\Sigma}^0_1\wedge C_p\cap C_q=\emptyset\text{ if }p\neq q\Big\}$.

Assume that α is suitable and $a_0,a_1\in\Delta^1_1$ satisfy $A_0\cap A_1=\emptyset$. Then Lemma 6.7.(b) gives $r(\overline{\alpha},a_0,a_1)$ and $r(\overline{\alpha},a_1,a_0)$, or $r\big((\alpha')_p,a_0,a_1\big)$. We set $R(\overline{\alpha},a_0,a_1):=\neg\mathcal{U}_{r(\overline{\alpha},a_0,a_1)}$ in the same fashion as before, and

$$R'(\alpha,a_0,a_1):=\begin{cases}\overline{R(\overline{\alpha},a_0,a_1)}^{\tau_1}\cap\overline{R(\overline{\alpha},a_1,a_0)}^{\tau_1}\text{ if we are in Case (1),}\\[2ex]\bigcap_{p\geq 1}\ \overline{R\big((\alpha')_p,a_0,a_1\big)}^{\tau_1}\text{ if we are in Case (2).}\end{cases}$$

We now give the self-dual version of Lemma 6.8.

Lemma 6.24 *Let α suitable, and $a_0,a_1\in\Delta^1_1$ such that $A_0\cap A_1=\emptyset$. We assume that $R'(\alpha,a_0,a_1)=\emptyset$. Then A_0 is separable from A_1 by a $\Delta^1_1\cap\Delta(\mathbf{\Gamma}_{c(\alpha)})(\tau_1)$ set.*

Proof. (1) As $\overline{R(\overline{\alpha},a_0,a_1)}^{\tau_1}\cap\overline{R(\overline{\alpha},a_1,a_0)}^{\tau_1}=\emptyset$, there is $C\in\mathbf{\Delta}^0_1(\tau_1)$ separating $R(\overline{\alpha},a_0,a_1)$ from $R(\overline{\alpha},a_1,a_0)$. As $R(\overline{\alpha},a_0,a_1)$ and $R(\overline{\alpha},a_1,a_0)$ are Σ^1_1, we may assume that $C\in\Delta^1_1$, by Theorem 4.2.2. A double application of Lemmas 6.7.(b) and 6.8 gives some sets $B_0,B_1\in\Delta^1_1\cap\mathbf{\Gamma}_{c(\overline{\alpha})}(\tau_1)$ such that B_0 (resp., B_1) separates $A_0\cap C$ (resp., $A_1\setminus C$) from $A_1\cap C$ (resp., $A_0\setminus C$). Now the set $(B_0\cap C)\cup(\neg B_1\cap\neg C)$ is suitable.

(2) The proof is similar, but we have to make some Δ^1_1-selection. As Θ^∞ is Π^1_1 and $r\big((\alpha')_p,a_0,a_1\big)$ is Δ^1_1 and completely determined by $(\alpha')_p$, a_0 and a_1, the sequence $\Big(r\big((\alpha')_p,a_0,a_1\big)\Big)_{p\geq 1}$ is Δ^1_1. As $\bigcap_{p\geq 1}\ \overline{R\big((\alpha')_p,a_0,a_1\big)}^{\tau_1}=\emptyset$, there is a Δ^1_1-recursive map $f:(\omega^\omega)^d\rightarrow\omega$ such that $f(\vec{\alpha})\geq 1$ and $\vec{\alpha}\notin\overline{R\big((\alpha')_{f(\vec{\alpha})},a_0,a_1\big)}^{\tau_1}$ for each $\vec{\alpha}\in(\omega^\omega)^d$.

We set $U_p:=f^{-1}(\{p\})$, so that U_p and $R\big((\alpha')_p,a_0,a_1\big)$ are disjoint Σ^1_1 sets and separable by a τ_1-open set. By Theorem 4.2.2, there is $V_p\in\Delta^1_1\cap\mathbf{\Sigma}^0_1(\tau_1)$ separating them. Moreover, we may assume that the sequence (V_p) is Δ^1_1. We reduce the sequence (V_p) into a Δ^1_1-sequence (C_p) of $\Delta^1_1\cap\mathbf{\Sigma}^0_1(\tau_1)$ sets. Note that (C_p) is a partition of $(\omega^\omega)^d$ into $\mathbf{\Delta}^0_1(\tau_1)$ sets. As $R\big((\alpha')_p,a_0,a_1\big)\cap C_p=\emptyset$, Lemma 6.8 gives $\beta',\gamma'\in\omega^\omega$ such that $\big((\alpha')_p,(\beta')_p,(\gamma')_p\big)\in\Upsilon^\infty$ and $C_{(\gamma')_p}$ separates $A_1\cap C_p$ from $A_0\cap C_p$ for each $p\geq 1$. Moreover, we may assume that $\beta',\gamma'\in\Delta^1_1$. Now the set $\bigcup_{p\geq 1}\ (\neg C_{(\gamma')_p}\cap C_p)$ is suitable. $\square$

We now give the self-dual version of Theorem 6.9.

Theorem 6.25 *Let T_d be a tree with Δ^1_1 suitable levels, α suitable, $\beta^\varepsilon,\gamma^\varepsilon\in\omega^\omega$ with $(\alpha,\beta^\varepsilon,\gamma^\varepsilon)\in\Upsilon^\infty_1$, $S^\varepsilon:=j_d^{-1}(C^{\omega^\omega}_{\gamma^\varepsilon})\cap\lceil T_d\rceil$, and $a_0,a_1,\underline{a}_0,\underline{a}_1,r\in\omega^\omega$ such that $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$. We assume that S^0 and S^1 are disjoint. Then one of the following holds:*

(a) $R'(\alpha,a_0,a_1)=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S^0,S^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Now we can state the version of Theorem 4.2.2 for the self-dual Wadge classes of Borel sets.

Theorem 6.26 *Let T_d be a tree with Δ^1_1 suitable levels, α suitable, $\beta^\varepsilon,\gamma^\varepsilon\in\omega^\omega$ with $(\alpha,\beta^\varepsilon,\gamma^\varepsilon)\in\Upsilon^\infty_1$, $S^\varepsilon:=j_d^{-1}(C^{\omega^\omega}_{\gamma^\varepsilon})\cap\lceil T_d\rceil$, and $a_0,a_1,\underline{a}_0,\underline{a}_1,r\in\omega^\omega$ such that $\vec{v}:=(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$. We assume that S^0, S^1 are disjoint and not separable by a pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set. Then the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set.

(c) The set A_0 is not separable from A_1 by a $\Delta(\mathbf{\Gamma}_{c(\alpha)})(\tau_1)$ set.

(d) $R'(\alpha,a_0,a_1)\neq\emptyset$.

(e) The inequality $\big((d^\omega)_{i\in d},S^0,S^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ holds.

Proof. We argue as in the proof of Theorem 6.10, using Lemma 6.24 (resp., Theorem 6.25) instead of Lemma 6.8 (resp., Theorem 6.9). $\square$

Proof of Theorem 1.8.(2). We argue as in the proof of Theorem 1.8.(1). Theorem 5.2.8 gives $\overline{u}$ or $\big((u')_p\big)_{p\geq 1}$. The equalities in Theorem 5.2.8 hold in ω^ω, but also in any 0-dimensional Polish space (we argue like in Lemma 5.2.2 to see it). Using Definition 5.1.2, we can build $u\in\mathcal{D}$ with $\mathbf{\Gamma}=\mathbf{\Gamma}_u$.

Using Lemmas 6.2 and 6.4, we get $\alpha\in\Lambda^\infty$ normalized with $\mathbf{\Gamma}_{c(\alpha)}=\mathbf{\Gamma}_u$, and $\overline{\alpha}\in\Lambda^\infty$ (or $\alpha'\in\Lambda^\infty$ such that $(\alpha')_p$ is) normalized with $\mathbf{\Gamma}_{\overline{u}}=\mathbf{\Gamma}_{c(\overline{\alpha})}$ (or $\mathbf{\Gamma}_{(u')_p}=\mathbf{\Gamma}_{c((\alpha')_p)}$).

By Theorem 4.1.3 in [Lo-SR2] there is $B^\varepsilon\in\mathbf{\Gamma}(\omega^\omega)$ with $S^\varepsilon=j_d^{-1}(B^\varepsilon)\cap\lceil T_d\rceil$. To simplify the notation, we may assume that T_d has Δ^1_1 levels, α, as well as $\overline{\alpha}$ (or α'), are Δ^1_1, and A_0, A_1 are Σ^1_1. By Lemma 6.5 there are $\beta^\varepsilon,\gamma^\varepsilon\in\omega^\omega$ such that $(\alpha,\beta^\varepsilon,\gamma^\varepsilon)\in\Upsilon^\infty_1$ and $C^{\omega^\omega}_{\gamma^\varepsilon}=B^\varepsilon$. Lemma 6.7.(b) gives $\underline{a}_0,\underline{a}_1,r$ with $(\alpha,a_0,a_1,\underline{a}_0,\underline{a}_1,r)\in\Theta^\infty$. Lemma 6.24 implies that $R'(\alpha,a_0,a_1)\neq\emptyset$. So (b) holds, by Theorem 6.26. $\square$

Proof of Theorem 6.25. (1) Let $C^\varepsilon_{\varepsilon'}\in\mathbf{\Sigma}^0_1(\lceil T_d\rceil)$, $A^\varepsilon_0\in\mathbf{\Gamma}_{c(\overline{\alpha})}(\lceil T_d\rceil)$, $A^\varepsilon_1\in\check{\mathbf{\Gamma}}_{c(\overline{\alpha})}(\lceil T_d\rceil)$ such that $S^\varepsilon=(A^\varepsilon_0\cap C^\varepsilon_0)\cup(A^\varepsilon_1\cap C^\varepsilon_1)$. We reduce the family $(C^0_0,C^0_1,C^1_0,C^1_1)$ into a family $(O^0_0,O^0_1,O^1_0,O^1_1)$ of open subsets of $\lceil T_d\rceil$. Note that $S^\varepsilon\subseteq T^\varepsilon:=(A^\varepsilon_0\cap O^\varepsilon_0)\cup(A^\varepsilon_1\cap O^\varepsilon_1)\cup(\neg A^{1-\varepsilon}_0\cap O^{1-\varepsilon}_0)\cup(\neg A^{1-\varepsilon}_1\cap O^{1-\varepsilon}_1)$. We will in fact ensure that $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},T^0,T^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ if (a) does not hold, which will be enough.

Subcase 1. $|(\alpha)_0|=0$.

We set $o^\varepsilon_{\varepsilon'}:=h[\lceil T_d\rceil\setminus O^\varepsilon_{\varepsilon'}]$, so that $o^\varepsilon_{\varepsilon'}\in\mathbf{\Pi}^0_1([\subseteq])$. We also set

$$D:=\{\vec{s}\in T_d\mid\vec{s}=\vec{\emptyset}\vee\forall(\varepsilon,\varepsilon')\in 2^2\ \ \exists\mathcal{B}\in o^\varepsilon_{\varepsilon'}\ \ \vec{s}\in\mathcal{B}\},$$

$$D^\varepsilon_{\varepsilon'}:=\{\vec{s}\in T_d\mid\vec{s}\neq\vec{\emptyset}\wedge\forall\mathcal{B}\in o^\varepsilon_{\varepsilon'}\ \ \vec{s}\notin\mathcal{B}\ \wedge\forall(\varepsilon'',\varepsilon''')\in 2^2\setminus\{(\varepsilon,\varepsilon')\}\ \ \exists\mathcal{B}\in o^{\varepsilon''}_{\varepsilon'''}\ \ \vec{s}\in\mathcal{B}\},$$

so that $(D,D^0_0,D^0_1,D^1_0,D^1_1)$ is a partition of T_d. The proof is very similar to the proof of Theorem 4.4.2 when $\xi=1$. The changes to make in the conditions (1)-(7) are as follows:

(4) $U_{\vec{s}}\subseteq R'(\alpha,a_0,a_1)=\overline{A_0}^{\tau_1}\cap\overline{A_1}^{\tau_1}$ if $\vec{s}\in D$,

(5) $U_{\vec{s}}\subseteq A_0$ if $\vec{s}\in D^0_1\cup D^1_0$,

(6) $U_{\vec{s}}\subseteq A_1$ if $\vec{s}\in D^0_0\cup D^1_1$,

(7) $(\vec{s},\vec{t}\in D\vee\vec{s},\vec{t}\in D^\varepsilon_{\varepsilon'})\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$.

We conclude as in the proof of Theorem 4.4.2.

Subcase 2. $|(\alpha)_0|\geq 1$.

We will have the same scheme of construction as in the proof of Theorem 6.9. As long as $\vec{t}\in D$, we will have $U_{\vec{t}}\subseteq R'(\alpha,a_0,a_1)$. If $\vec{t}\in D^\varepsilon_{\varepsilon'}$, then all the extensions of $\vec{t}$ will stay in $D^\varepsilon_{\varepsilon'}$, and we will copy the construction of the proof of Theorem 6.9, since inside the clopen set defined by $\vec{t}$ we want to reduce a pair $(\tilde{S}^0,\tilde{S}^1)$ to (A_0,A_1).

As $A^\varepsilon_0\in\mathbf{\Gamma}_{c(\overline{\alpha})}(\lceil T_d\rceil)$, there is $B^\varepsilon_0\in\mathbf{\Gamma}_{c(\overline{\alpha})}(\omega^\omega)$ with $A^\varepsilon_0=j_d^{-1}(B^\varepsilon_0)\cap\lceil T_d\rceil$. As $\overline{\alpha}\in\Delta^1_1\cap\Lambda^\infty$, Lemma 6.5.(b) gives $\beta^\varepsilon_0,\gamma^\varepsilon_0\in\omega^\omega$ such that $(\overline{\alpha},\beta^\varepsilon_0,\gamma^\varepsilon_0)\in\Upsilon^\infty_1$ and $C^{\omega^\omega}_{\gamma^\varepsilon_0}=B^\varepsilon_0$. Similarly, there are $\beta^\varepsilon_1,\gamma^\varepsilon_1\in\omega^\omega$ such that $(\overline{\alpha},\beta^\varepsilon_1,\gamma^\varepsilon_1)\in\Upsilon^\infty_1$ and $A^\varepsilon_1=j_d^{-1}(\neg C^{\omega^\omega}_{\gamma^\varepsilon_1})\cap\lceil T_d\rceil$.

We can associate with any $(\varepsilon,\varepsilon')\in 2^2$ the objects met before, among which the function $\mathcal{Z}^{\varepsilon,\varepsilon'}$, the ordinals $\eta_s^{\varepsilon,\varepsilon'}$, the resolution families $(R^{(\rho)}_{\varepsilon,\varepsilon',s})_{\rho\leq\eta_s^{\varepsilon,\varepsilon'}}$, the ordinals $\rho(\varepsilon,\varepsilon',s,\vec{s}\,)$. Instead of considering the set $P_q(s)$, we will consider $P_q^{\varepsilon,\varepsilon'}(s)\cap D^\varepsilon_{\varepsilon'}$. If $\vec{t}\in D^\varepsilon_{\varepsilon'}$, then we set $\vec{w}(\vec{t}\,):=\vec{w}^\varepsilon_{\varepsilon'}$. This allows us to define $s(\vec{t}\,)\in\mathfrak{T}\big(\vec{w}(\vec{t}\,)\big)$ and $S(\vec{t}\,)\in\mathcal{M}_{\vec{w}(\vec{t})}$. We also set

$$\vec{v}(\vec{t}\,):=\left\{\begin{array}{l}(\overline{\alpha},a_0,a_1,\underline{a}_0,\underline{a}_1,r)\text{ if }\vec{t}\in D^0_0\cup D^1_1,\\ \\ (\overline{\alpha},a_1,a_0,\underline{a}_0,\underline{a}_1,r)\text{ if }\vec{t}\in D^0_1\cup D^1_0.\end{array}\right.$$

The other modifications to make in the conditions (1)-(6) are as follows. In condition (4), we ask for the inclusion $U_{\vec{t}}\subseteq R\big(S(\vec{t}\,)\big)$ only if $\vec{t}\notin D$. If $\vec{t}\in D$, then we want that $U_{\vec{t}}\subseteq R'(\alpha,a_0,a_1)$. Condition (6) was described when $\vec{s},\vec{t}\in D^\varepsilon_{\varepsilon'}$. If $\vec{s},\vec{t}\in D$, then we also want that $U_{\vec{t}}\subseteq U_{\vec{s}}$.

The sequence $\mathcal{F}(\vec{\beta})$ is defined if $\beta\in C^0_0\cup C^0_1\cup C^1_0\cup C^1_1$. If $\beta\notin C^0_0\cup C^0_1\cup C^1_0\cup C^1_1$, then $\vec{\beta}|k\in D$ for each integer k, and $\mathcal{F}(\vec{\beta})$ is also defined. The definition of $\vec{v}(\vec{t}\,)$ ensures that $T^\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon)$.

The defintion of $\Psi(\overrightarrow{tm})$ is done if $\overrightarrow{tm}\notin D$. If $\overrightarrow{tm}\in D$, then we simply set

$$\Psi(\overrightarrow{tm}):=U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i).$$

Then we conclude as in the proof of Theorem 6.9.

(2) Let $C^\varepsilon_p\in\mathbf{\Sigma}^0_1(\lceil T_d\rceil)$ and $A^\varepsilon_p\in\mathbf{\Gamma}_{c((\alpha')_p)}(\lceil T_d\rceil)$ such that $S^\varepsilon=\bigcup_{p\geq 1}\ (A^\varepsilon_p\cap C^\varepsilon_p)$. We reduce the family $(C^0_1,C^0_2,...,C^1_1,C^1_2,...)$ into a family $(O^0_1,O^0_2,...,O^1_1,O^1_2,...)$ of open subsets of $\lceil T_d\rceil$. Note that $S^\varepsilon\subseteq T^\varepsilon:=(A^\varepsilon_1\cap O^\varepsilon_1)\cup\bigcup_{p\geq 1}\ \big((\neg A^{1-\varepsilon}_p\cap O^{1-\varepsilon}_p)\cup(A^\varepsilon_{p+1}\cap O^\varepsilon_{p+1})\big)$. We will in fact ensure that $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},T^0,T^1\big)\leq\big((\omega^\omega)_{i\in d},A_0,A_1\big)$ if (a) does not hold, which will be enough.

The proof is similar. We can assume that $\left|\big((\alpha')_p\big)_0\right|\geq 1$ for each $p\geq 1$, since $(\mathbf{\Gamma}_{c((\alpha')_p)})_{p\geq 1}$ is strictly increasing. So there is no Subcase 1. We set

$$\vec{v}(\vec{t}\,):=\left\{\begin{array}{l}(\overline{\alpha},a_0,a_1,\underline{a}_0,\underline{a}_1,r)\text{ if }\vec{t}\in\bigcup_{p\geq 1}\ D^0_p,\\ \\ (\overline{\alpha},a_1,a_0,\underline{a}_0,\underline{a}_1,r)\text{ if }\vec{t}\in\bigcup_{p\geq 1}\ D^1_p.\end{array}\right.$$

We conclude as in Case 1. □

7 Injectivity complements

In the introduction, we saw that G. Debs proved that we can have the f_i's one-to-one in Theorem 1.3 when $d=2$, $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi\}$ and $\xi\geq 3$.

• This cannot be extended to higher dimensions, even if we replace $(d^\omega)^d$ with $\Pi_{i\in d}\ Z_i$, where Z_i is a sequence of Polish spaces.

Indeed, we argue by contradiction. Recall the proof of Theorem 3.1. We saw that there is C_ξ in $\mathbf{\Sigma}^0_\xi(2^\omega)\backslash\mathbf{\Pi}^0_\xi$ such that $\mathbb{S}^3_\xi:=\{\vec{\alpha}\in\lceil T_3\rceil \mid \mathcal{S}(\alpha_0\Delta\alpha_1)\in C_\xi\}$ is not separable from $\lceil T_3\rceil\backslash\mathbb{S}^3_\xi$ by a pot($\mathbf{\Pi}^0_\xi$) set. We set

$$\begin{aligned} B^0 &:= \{\vec{\alpha}\in 3^\omega\times 3^\omega\times 1 \mid \mathcal{S}(\alpha_0\Delta\alpha_1)\in C_\xi\},\\ B^1 &:= \{\vec{\alpha}\in 3^\omega\times 1\times 3^\omega \mid \mathcal{S}(\alpha_0\Delta\alpha_2)\in C_\xi\},\\ B^2 &:= \{\vec{\alpha}\in 1\times 3^\omega\times 3^\omega \mid \mathcal{S}(\alpha_1\Delta\alpha_2)\in C_\xi\}.\end{aligned}$$

Let $O:3^\omega\to 1$. As $\mathbb{S}^3_\xi:=(\text{Id}_{3^\omega}\times\text{Id}_{3^\omega}\times O)^{-1}(B^0)\cap\lceil T_3\rceil$, $B^0\notin$ pot($\mathbf{\Pi}^0_\xi$). Similarly, $B^1, B^2\notin$ pot($\mathbf{\Pi}^0_\xi$). This implies that the Z_i's have cardinality at most one, and $\mathbb{S}_0\in\mathbf{\Delta}^0_1$. Thus $\mathbb{S}_0$ is separable from $\mathbb{S}_1$ by a pot($\mathbf{\Pi}^0_\xi$) set, which is absurd.

• If $d=\omega$, $\mathbf{\Gamma}=\mathbf{\Pi}^0_\xi$ and $\xi\geq 3$, then we cannot ensure that at least two of the f_i's are one-to-one. Indeed, we again argue by contradiction. Consider $X_i:=\omega$, and $B_\xi\in\mathbf{\Sigma}^0_\xi(\omega^\omega)\setminus\mathbf{\Pi}^0_\xi$. Then B_ξ is not pot($\mathbf{\Pi}^0_\xi$) since the topology on ω is discrete. This implies that two of the Z_i's at least are countable, say Z_0, Z_1 for example. Consider now $A_0:=\mathbb{S}^\omega_\xi$ and $A_1:=\lceil T_\omega\rceil\setminus\mathbb{S}^\omega_\xi$. Then $(f_i\circ\Pi_i)[\mathbb{S}_0]$ is countable for each $i\in 2$. Thus $P:=(\Pi_{i\in d}\ f_i)[\mathbb{S}_0]\subseteq\mathbb{S}^\omega_\xi\subseteq\lceil T_\omega\rceil$ is countable since an element of $\lceil T_\omega\rceil$ is completely determined by two of its coordinates. Thus $P\in$ pot($\mathbf{\Sigma}^0_2$) $\subseteq$ pot($\mathbf{\Pi}^0_\xi$). Therefore $(\Pi_{i\in d}\ f_i)^{-1}(P)$ is a pot($\mathbf{\Pi}^0_\xi$) set separating $\mathbb{S}_0$ from $\mathbb{S}_1$, which is absurd.

• However, if $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi,\mathbf{\Delta}^0_\xi\}$ and $\xi\geq 3$, then we can ensure that $(\Pi_{i\in d}\ f_i)_{|\mathbb{S}_0\cup\mathbb{S}_1}$ is one-to-one, using G. Debs's proof and some additional arguments. This is also true if $\mathbf{\Gamma}=\mathbf{\Gamma}_u$ is a non self-dual Wadge class of Borel sets with $u(0)\geq 3$. This leads to the following notation. Let $(Z_i)_{i\in d}$, $(X_i)_{i\in d}$ be sequences of Polish spaces, and S_0, S_1 (resp., A_0, A_1) disjoint analytic subsets of $\Pi_{i\in d}\ Z_i$ (resp., $\Pi_{i\in d}\ X_i$). Then

$$\begin{aligned}\big((Z_i)_{i\in d},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in d},A_0,A_1\big)\ \Leftrightarrow\ &\forall i\in d\ \exists f_i:Z_i\to X_i\ \text{ continuous such that}\\ &(\Pi_{i\in d}\ f_i)_{|S_0\cup S_1}\ \text{ is one-to-one and }\ \forall\varepsilon\in 2\ \ S_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon).\end{aligned}$$

Theorem 7.1 *There is no tuple $\big((Z_i)_{i\in 2},S_0,S_1\big)$, where the Z_i's are Polish spaces and S_0, S_1 disjoint analytic subsets of $\Pi_{i\in 2}\ Z_i$, such that for any tuple $\big((X_i)_{i\in 2},B_0,B_1\big)$ of the same type exactly one of the following holds:*

(a) The set B_0 is separable from B_1 by a pot($\mathbf{\Pi}^0_1$) set.

(b) The inequality $\big((Z_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},B_0,B_1\big)$ holds.

One can prove this result using the Borel digraph $B_0:=\bigcup_{n\in\omega}\ \text{Gr}(g_{n|2^\omega\setminus M})$ considered in [L5] (see Section 3), which has countable vertical sections but is not locally countable. We give here another proof which moreover shows that we cannot hope for a positive result, even if B_0 is locally countable. This has to be noticed, since the locally countable sets have been considered a lot in the last decades.

Lemma 7.2 *Let $\mathbf{\Gamma}$ be a Borel class, and $\big((Z_i)_{i\in 2},S_0,S_1\big)$ as in the statement of Theorem 5.1 such that S_0 is not separable from S_1 by a pot($\mathbf{\Gamma}$) set. Then $S_0\cap(\Pi''_0S_1\times\Pi''_1S_1)$ is not separable from S_1 by a pot($\mathbf{\Gamma}$) set. Moreover, S_0 is not separable from $S_1\cap(\Pi''_0S_0\times\Pi''_1S_0)$ by a pot($\mathbf{\Gamma}$) set.*

Proof. We prove the first assertion by contradiction, which gives $P \in \text{pot}(\mathbf{\Gamma})$. The first reflection theorem gives Borel sets C_0, C_1 such that $\Pi_i'' S_1 \subseteq C_i$ and $S_0 \cap (C_0 \times C_1) \subseteq P$. Now

$$S_0 \subseteq P \cup (\neg C_0 \times Z_1) \cup (Z_0 \times \neg C_1) \subseteq \neg S_1,$$

which contradicts the fact that S_0 is not separable from S_1 by a pot$(\mathbf{\Gamma})$ set.

We prove the second assertion using the first one, passing to complements. $\square$

Lemma 7.3 *Let $\big((Z_i)_{i\in 2}, S_0, S_1\big)$ and $\big((X_i)_{i\in 2}, B_0, B_1\big)$ be as in the statement of Theorem 5.1 such that $\big((Z_i)_{i\in 2}, S_0, S_1\big) \sqsubseteq \big((X_i)_{i\in 2}, B_0, B_1\big)$, $(f_i)_{i\in 2}$ witnesses for this inequality, and $\varepsilon_0 \in 2$ such that B_{ε_0} is Borel locally countable. Then $f_{i|\Pi_i'' S_{\varepsilon_0}}$ is countable-to-one for each $i \in 2$ and S_{ε_0} is locally countable.*

Proof. The inequality $\big((Z_i)_{i\in 2}, S_0, S_1\big) \sqsubseteq \big((X_i)_{i\in 2}, B_0, B_1\big)$ gives $f_i : Z_i \to X_i$ continuous such that $(\Pi_{i\in 2}\ f_i)_{|S_0\cup S_1}$ is one-to-one, and also $S_\varepsilon \subseteq (\Pi_{i\in 2}\ f_i)^{-1}(B_\varepsilon)$ for each $\varepsilon \in 2$.

• By the Lusin-Novikov theorem and Lemma 2.4.(a) in [L2] we can find Borel one-to-one partial functions b_n with Borel domain such that $B_{\varepsilon_0} = \bigcup_{n\in\omega}\ \text{Gr}(b_n)$. Let us prove that

$$f_{i|\Pi_i[S_{\varepsilon_0} \cap (\Pi_{i\in 2}\ f_i)^{-1}(\text{Gr}(b_n))]}$$

is one-to-one for each $i \in 2$.

Assume for example that $i = 0$. Let $z \neq z' \in \Pi_0\big[S_{\varepsilon_0} \cap (\Pi_{i\in 2}\ f_i)^{-1}\big(\text{Gr}(b_n)\big)\big]$, and $y, y' \in Z_1$ such that $(z,y), (z',y') \in S_{\varepsilon_0} \cap (\Pi_{i\in 2}\ f_i)^{-1}\big(\text{Gr}(b_n)\big)$. As $(z,y) \neq (z',y')$, we get

$$\big(f_0(z), f_1(y)\big) \neq \big(f_0(z'), f_1(y')\big).$$

But $b_n\big(f_0(z)\big) = f_1(y)$, $b_n\big(f_0(z')\big) = f_1(y')$, so that $f_0(z) \neq f_0(z')$ since b_n is a partial function. If $i = 1$, then we use the fact that b_n is one-to-one to see that $f_{i|\Pi_i[S_{\varepsilon_0} \cap (\Pi_{i\in 2}\ f_i)^{-1}(\text{Gr}(b_n))]}$ is also one-to-one.

• This proves that $f_{i|\Pi_i'' S_{\varepsilon_0}}$ is countable-to-one since $S_{\varepsilon_0} = \bigcup_{n\in\omega}\ S_{\varepsilon_0} \cap (\Pi_{i\in 2}\ f_i)^{-1}\big(\text{Gr}(b_n)\big)$.

• Now S_{ε_0} is locally countable since $S_{\varepsilon_0} \subseteq (\Pi_{i\in 2}\ f_{i|\Pi_i'' S_{\varepsilon_0}})^{-1}(B_{\varepsilon_0})$, B_{ε_0} is locally countable and $f_{i|\Pi_i'' S_{\varepsilon_0}}$ is countable-to-one for each $i \in 2$. $\square$

Lemma 7.4 *Let Y be a Polish space, C a Borel subset of Y and $(m_n)_{n\in\omega}$ a sequence of Borel partial functions from a Borel subset of C into C. We assume that $M := \bigcup_{n\in\omega}\ Gr(m_n)$ is disjoint from $\Delta(C)$, but not separable from $\Delta(C)$ by a pot$(\mathbf{\Pi}^0_1)$ set. Then there are integers $n < p$ and $y \in C$ such that $m_n(y)$ and $m_n\big(m_p(y)\big)$ are defined.*

Proof. We may assume that Y is recursively presented and C, M and the m_n's are Δ^1_1. We put

$$V := \bigcup \{D \in \Delta^1_1(Y) \mid D^2 \cap M \text{ has finite vertical sections}\}.$$

Then $V \in \Pi^1_1(Y)$.

Case 1. $V = Y$.

We can find a sequence $(D_n)_{n\in\omega}$ of Δ^1_1 subsets of Y such that $Y = \bigcup_{n\in\omega}\ D_n$ and $D^2_n \cap M$ has finite vertical sections. By Theorem 3.6 in [Lo2], $D^2_n \cap M$ is pot($\mathbf{\Pi}^0_1$), so that $D^2_n \setminus M$ is pot($\mathbf{\Sigma}^0_1$). Thus $\Delta(C) \subseteq \bigcup_{n\in\omega}\ D^2_n \setminus M \subseteq \neg M$ and $\Delta(C)$ is separable from M by a pot($\mathbf{\Sigma}^0_1$) set, which is absurd.

Case 2. $V \neq Y$.

The first reflection theorem proves that for each nonempty Σ^1_1 subset S of Y contained in $Y \setminus V$ there is $y \in S$ such that $(S^2 \cap M)_y$ is infinite. So there is an integer n such that $(Y \setminus V)^2 \cap \text{Gr}(m_n) \neq \emptyset$. In particular, $S := (Y \setminus V) \cap m^{-1}_n(Y \setminus V)$ is a nonempty Σ^1_1 subset of Y, which gives $y \in S$ such that $(S^2 \cap M)_y$ is infinite. This proves the existence of $p > n$ such that $\big(y, m_p(y)\big) \in S^2$. Note that $y \in C$ since $Y \setminus C \subseteq V$. Now it is clear that n, p and y are suitable. $\square$

Lemma 7.5 *Let $i \in 2$, Y_i a Polish space, δ_i a Borel subset of Y_i, $c : \delta_0 \to \delta_1$ a Borel isomorphism, $n \in \omega$, c_n a Borel one-to-one partial function from Y_0 into Y_1 with Borel domain, and $C_0 := \bigcup_{n\in\omega}\ Gr(c_n)$. We assume that $C_0 \cap (\delta_0 \times \delta_1)$ is disjoint from $Gr(c)$, but not separable from $Gr(c)$ by a pot($\mathbf{\Pi}^0_1$) set. Then there are integers $n < p$ and $y_0 \in Y_0$ such that $(cc^{-1}_n c_p)(y_0)$ and $(cc^{-1}_n c)(y_0)$ are defined and different.*

Proof. We set $c'_n := c_{n|\delta_0 \cap c^{-1}_n(\delta_1)}$, so that $C_0 \cap (\delta_0 \times \delta_1) = \bigcup_{n\in\omega}\ \text{Gr}(c'_n)$. Now we consider the pre-images

$$\Delta(\delta_1) = (c^{-1} \times \text{Id}_{\delta_1})^{-1}\big(\text{Gr}(c)\big)$$

and $\text{Gr}(c''_n) = (c^{-1} \times \text{Id}_{\delta_1})^{-1}\big(\text{Gr}(c'_n)\big)$, where $c''_n := c'_n \circ c^{-1}_{|c[\delta_0 \cap c^{-1}_n(\delta_1)]}$. Note that c''_n is a Borel one-to-one partial function with Borel domain and that $C''_0 := \bigcup_{n\in\omega}\ \text{Gr}(c''_n)$ is not separable from $\Delta(\delta_1)$ by a pot($\mathbf{\Pi}^0_1$) set. This implies that $\bigcup_{n\in\omega}\ \text{Gr}\big((c''_n)^{-1}\big)$ is not separable from $\Delta(\delta_1)$ by a pot($\mathbf{\Pi}^0_1$) set.

By Lemma 7.4 there are integers $n < p$ and $y_1 \in \delta_1$ such that $(c''_n)^{-1}(y_1)$ and $(c''_n)^{-1}\big((c''_p)^{-1}(y_1)\big)$ are defined. We set $y_0 := (c'_p)^{-1}(y_1)$, so that $\big(c(c'_n)^{-1}c'_p\big)(y_0)$ and $\big(c(c'_n)^{-1}c\big)(y_0)$ are defined and equal respectively to $\big(cc^{-1}_n c_p\big)(y_0)$ and $\big(cc^{-1}_n c\big)(y_0)$. Now note that $y_1 \neq (c''_p)^{-1}(y_1)$ for each y_1 in the range of c''_p. This implies that $(c''_n)^{-1}(y_1) \neq (c''_n)^{-1}\big((c''_p)^{-1}(y_1)\big)$,

$$\big(c(c'_n)^{-1}\big)(y_1) \neq \big(c(c'_n)^{-1}c(c'_p)^{-1}\big)(y_1),$$

$\big(c(c'_n)^{-1}c'_p\big)(y_0) \neq \big(c(c'_n)^{-1}c\big)(y_0)$ and $\big(cc^{-1}_n c_p\big)(y_0) \neq \big(cc^{-1}_n c\big)(y_0)$. $\square$

Lemma 7.6 *Let Y be a Polish space, $n \in \omega$, c and c_n continuous open partial functions from Y into Y with open domain, $\varepsilon \in 2$, $C^\varepsilon := \bigcup_{n\in\omega}\ Gr(c_{2n+\varepsilon})$. We assume that C^0 is disjoint from $C^1 \cup Gr(c)$, but $\emptyset \neq Gr(c) \subseteq \overline{C^0} \cap \overline{C^1}$. Then C^0 is not separable from C^1 by a pot($\mathbf{\Delta}^0_1$) set, and C^0 is not separable from $Gr(c)$ by a pot($\mathbf{\Pi}^0_1$) set. If moreover the domains $Dom(c_n)$ are dense, then $C^0 \cap (\bigcap_{n\in\omega}\ Dom(c_n) \times 2^\omega)$ is not separable from $C^1 \cap (\bigcap_{n\in\omega}\ Dom(c_n) \times 2^\omega)$ by a pot($\mathbf{\Delta}^0_1$) set.*

Proof. We argue by contradiction, which gives $P\in\text{pot}(\mathbf{\Delta}^0_1)$. Let G_i be a dense G_δ subset of Y_i such that $P\cap (G_0\times G_1)\in \mathbf{\Delta}^0_1(G_0\times G_1)$. The proof of Lemma 3.5 in [L1] shows the inclusion $\text{Gr}(c)\subseteq\overline{\text{Gr}(c)\cap (G_0\times G_1)}$, and similarly with c_n. Thus

$$\text{Gr}(c)\subseteq\overline{\overline{C^0}\cap\overline{C^1}\cap (G_0\times G_1)}\subseteq\overline{\overline{C^0\cap (G_0\times G_1)}\cap\overline{C^1\cap (G_0\times G_1)}\cap (G_0\times G_1)}$$

$$\subseteq\overline{\big(P\cap (G_0\times G_1)\big)\setminus\big(P\cap (G_0\times G_1)\big)}=\emptyset,$$

which is absurd. The last assertion follows since we may assume that $G_0\subseteq\bigcap_{n\in\omega}\ \text{Dom}(c_n)$. The proof of the second assertion is similar and simpler. $\square$

Lemma 7.7 *There is a tuple* $\big((Y_i)_{i\in 2},C_0,C_1\big)$ *such that*

(a) Y_0 *and* Y_1 *are Polish spaces.*

(b) $C_0=\bigcup_{n\in\omega}\ Gr(c_n)\subseteq\Pi_{i\in 2}\ Y_i$*, for some Borel one-to-one partial functions* c_n *with Borel domain.*

(c) $C_1=Gr(c)$*, for some Borel function* $c:Y_0\to Y_1$*.*

(d) C_0 *is disjoint from* C_1*, but not separable from* C_1 *by a pot(*$\mathbf{\Pi}^0_1$*) set.*

(e) We set $C^\varepsilon_0:=\big(\bigcup_{n\in\omega}\ Gr(c_{2n+\varepsilon})\big)\cap(\bigcap_{n\in\omega}\ Dom(c_n)\times 2^\omega)$*, for* $\varepsilon\in 2$*. Then* C^0_0 *is disjoint from* C^1_0*, but not separable from* C^1_0 *by a pot(*$\mathbf{\Delta}^0_1$*) set, and* $\overline{C^0_0}\cap\overline{C^1_0}\cap(\bigcap_{n\in\omega}\ Dom(c_n)\times 2^\omega)\subseteq Gr(c)$*.*

(f) The equality $\big(cc_n^{-1}c_p\big)(y_0)=\big(cc_n^{-1}c\big)(y_0)$ *holds as soon as the two members of the equality are defined and* $n<p$*.*

Proof. We set $Y_i:=2^\omega$ and $c(\alpha)(k):=\alpha(2k)$.

• We first build an increasing sequence $(S_n)_{n\in\omega}$ of co-infinite subsets of ω, a sequence $(\psi_n)_{n\in\omega}$ of bijections, and a sequence $(h_n)_{n\in\omega}$ of homeomorphisms of 2^ω onto itself. We do it by induction on n. We set $S_0:=\emptyset$, $\psi_0:=\text{Id}_\omega$ and $h_0:=\text{Id}_{2^\omega}$. Assume that $(S_q)_{q\leq n}$, $(\psi_q)_{q\leq n}$ and $(h_q)_{q\leq n}$ are constructed, which is the case for $n=0$. We define a map $\varphi_n:\omega\to\omega$ by

$$\varphi_n(k):=\begin{cases}\psi_n^{-1}(k)\text{ if }k\notin 2S_n,\\ \\ \frac{k}{2}\text{ if }k\in 2S_n.\end{cases}$$

Note that φ_n is a bijection. We set $S_{n+1}:=\varphi_n[2\omega]\cup(n+1)$, which is co-infinite. The sequence $(S_n)_{n\in\omega}$ is increasing since $S_n=\varphi_n[2S_n]\subseteq S_{n+1}$. As S_{n+1} is co-infinite we can build the bijection $\psi_{n+1}:\omega\setminus S_{n+1}\to\omega\setminus 2S_{n+1}$ in such a way that $\psi_{n+1}(k)\neq\psi_q(k)$ for infinitely many $k\notin S_{n+1}$, for each $q\leq n$. We set

$$h_{n+1}(\alpha)(k):=\begin{cases}c(\alpha)(k)\text{ if }k\in S_{n+1},\\ \\ \alpha\big(\psi_{n+1}(k)\big)\text{ if }k\notin S_{n+1}.\end{cases}$$

As h_{n+1} permutes the coordinates, it is an homeomorphism.

• We set $D_n:=\{\alpha\in 2^\omega\mid c(\alpha)\neq h_n(\alpha)\wedge\forall q<n\ \ h_n(\alpha)\neq h_q(\alpha)\}$, so that D_n is an open subset of 2^ω. We set $c_n:=h_{n|D_n}$, so that c_n is an homeomorphism from D_n onto its open range, C_0 is disjoint from C_1, and C^0_0 is disjoint from C^1_0.

Let us prove that D_n is dense for each integer n. Note that $D_0=\{\alpha\in 2^\omega\mid\exists k\in\omega\ \ \alpha(2k)\neq\alpha(k)\}$, which is clearly dense. Now D_{n+1} contains

$$\{\alpha\in 2^\omega\mid\exists k\notin S_{n+1}\ \ \alpha(2k)\neq\alpha\big(\psi_{n+1}(k)\big)\}\cap\bigcap_{q<n}\{\alpha\in 2^\omega\mid\exists k\notin S_{n+1}\ \ \alpha\big(\psi_{n+1}(k)\big)\neq\alpha\big(\psi_q(k)\big)\}.$$

The set $\{\alpha\in 2^\omega\mid\exists k\notin S_{n+1}\ \ \alpha(2k)\neq\alpha\big(\psi_{n+1}(k)\big)\}$ is open dense since the odd integers are in $\psi_{n+1}[\omega\setminus S_{n+1}]$. The set $\{\alpha\in 2^\omega\mid\exists k\notin S_{n+1}\ \ \alpha\big(\psi_{n+1}(k)\big)\neq\alpha\big(\psi_q(k)\big)\}$ is open dense by construction of ψ_{n+1}. This proves that D_{n+1} is dense.

• Note that $\text{Gr}(c)\subseteq\overline{C_0^0}\cap\overline{C_0^1}$ since $c(\alpha)|n=h_n(\alpha)|n$, D_n is dense and c is continuous. Lemma 7.6 proves the non-separation assertions. We also have $\overline{C_0^0}\cap\overline{C_0^1}\cap(\bigcap_{n\in\omega}\ \text{Dom}(c_n)\times 2^\omega)\subseteq\text{Gr}(c)$ since $c(\alpha)|n=h_n(\alpha)|n$ and c_n is continuous.

• Now it is enough to prove that $ch_n^{-1}h_p=ch_n^{-1}c$ if $n<p$. We have

$$h_n^{-1}(\beta)(j):=\begin{cases}\beta(k)\text{ if }j=2k\in 2S_n,\\[2ex]\beta\big(\psi_n^{-1}(j)\big)\text{ if }j\notin 2S_n.\end{cases}$$

Thus

$$(ch_n^{-1}c)(\alpha)(k)=c\big((h_n^{-1}c)(\alpha)\big)(k)=(h_n^{-1}c)(\alpha)(2k)=\begin{cases}c(\alpha)(k)\text{ if }k\in S_n,\\[2ex]c(\alpha)\big(\psi_n^{-1}(2k)\big)\text{ if }k\notin S_n.\end{cases}$$

Similarly,

$$(ch_n^{-1}h_p)(\alpha)(k)=\begin{cases}h_p(\alpha)(k)\text{ if }k\in S_n,\\[2ex]h_p(\alpha)\big(\psi_n^{-1}(2k)\big)\text{ if }k\notin S_n.\end{cases}$$

Note that $S_n=\varphi_n[2S_n]\subseteq S_{n+1}$, so that $S_n\subseteq S_p$. Thus $(ch_n^{-1}h_p)(\alpha)(k)=(ch_n^{-1}c)(\alpha)(k)$ if $k\in S_n$. If $k\notin S_n$, then $2k\notin 2S_n$ and $\varphi_n(2k)=\psi_n^{-1}(2k)\in S_{n+1}\subseteq S_p$. Thus

$$(ch_n^{-1}h_p)(\alpha)(k)=h_p(\alpha)\big(\psi_n^{-1}(2k)\big)=c(\alpha)\big(\psi_n^{-1}(2k)\big)=(ch_n^{-1}c)(\alpha)(k).$$

This finishes the proof. □

Proof of Theorem 7.1. We argue by contradiction. Note that S_0 is not separable from S_1 by a pot($\mathbf{\Pi}^0_1$) set since (b) holds. By Lemma 7.2 we may assume that the inequality $S_1\subseteq\Pi''_0S_0\times\Pi''_1S_0$ holds.

• Recall the digraph A_1 in [L5]. If we take $X_i:=2^\omega$, $B_0:=A_1$ and $B_1:=\Delta(2^\omega)$, then by Corollary 12 in [L5], B_0 is Borel locally countable, not pot($\mathbf{\Pi}^0_1$), and $B_1=\overline{B_0}\setminus B_0$. It follows that B_0 is not separable from B_1 by a pot($\mathbf{\Pi}^0_1$) set Q, since otherwise we would have $B_0=Q\cap\overline{B_0}\in$ pot($\mathbf{\Pi}^0_1$). This implies that $\big((X_i)_{i\in 2},B_0,B_1\big)$ satisfies condition (b) in Theorem 7.1. By Lemma 7.3, $f_{i|\Pi''_iS_0}$ is countable-to-one for each $i\in 2$ and S_0 is locally countable.

• Lemma 7.7 gives a tuple $\big((Y_i)_{i\in 2},C_0,C_1\big)$. Note that $\big((Y_i)_{i\in 2},C_0,C_1\big)$ satisfies condition (b) in Theorem 7.1, which gives $g_i:Z_i\rightarrow Y_i$. Lemma 7.3 implies that $g_{i|\Pi''_iS_0}$ is countable-to-one for each $i\in 2$. The first reflection theorem gives a Borel set $O_i\supseteq\Pi''_iS_0$ such that $f_{i|O_i}$ and $g_{i|O_i}$ are countable-to-one, for each $i\in 2$. By Lemma 2.4.(a) in [L2] we can find a partition $(O^i_n)_{n\in\omega}$ of O_i into Borel sets such that $f_{i|O^i_n}$ and $g_{i|O^i_n}$ are one-to-one, for each $i\in 2$.

• We set $S''_\varepsilon:=(\Pi_{i\in 2}\ f_{i|O_i})^{-1}(B_\varepsilon)\cap(\Pi_{i\in 2}\ g_i)^{-1}(C_\varepsilon)$, for each $\varepsilon\in 2$, so that S''_ε is a Borel subset of $\Pi_{i\in 2}\ Z_i$ containing S_ε. In particular, S''_0 is not separable from S''_1 by a pot$(\mathbf{\Pi}^0_1)$ set. We choose integers n_0 and n_1 such that $S''_0\cap(\Pi_{i\in 2}\ O^i_{n_i})$ is not separable from $S''_1\cap(\Pi_{i\in 2}\ O^i_{n_i})$ by a pot$(\mathbf{\Pi}^0_1)$ set. We set $D_\varepsilon:=(\Pi_{i\in 2}\ f_{i|O^i_{n_i}})[S''_\varepsilon\cap(\Pi_{i\in 2}\ O^i_{n_i})]$, so that D_0 is a Borel subset of B_0 which is not separable from D_1 by a pot$(\mathbf{\Pi}^0_1)$ set. Note that D_1 is a Borel subset of $B_1=\Delta(2^\omega)$. In particular, there is a Borel subset D of 2^ω such that $D_1=\Delta(D)$. By Lemma 7.2, $D_0\cap D^2$ is not separable from D_1 by a pot$(\mathbf{\Pi}^0_1)$ set. Let $h_i:D\rightarrow Y_i$ be defined by $h_i(\alpha):=(g_i\circ f^{-1}_{i|O^i_{n_i}})(\alpha)$. Then h_i is Borel, one-to-one, and $D_\varepsilon\cap D^2\subseteq B_\varepsilon\cap(\Pi_{i\in 2}\ h_i)^{-1}(C_\varepsilon)$.

• Note that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]$ is a Borel subset of C_1, which proves the existence of a Borel subset δ of Y_0 such that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]=\text{Gr}(c_{|\delta})$. If $y\neq y'\in\delta$, then $\big(y,c(y)\big)=\big(h_0(d),h_1(d)\big)$ and

$$\big(y',c(y')\big)=\big(h_0(d'),h_1(d')\big)$$

for some $d\neq d'\in D$. As h_1 is one-to-one we get $c(y)\neq c(y')$, $c_{|\delta}$ is one-to-one and $c''\delta$ is Borel.

As $D_0\cap D^2\subseteq(\Pi_{i\in 2}\ h_i)^{-1}(C_0)$ and $D_1\subseteq(\Pi_{i\in 2}\ h_i)^{-1}\big(\text{Gr}(c_{|\delta})\big)$, C_0 is not separable from $\text{Gr}(c_{|\delta})$ by a pot$(\mathbf{\Pi}^0_1)$ set. By Lemma 7.2, $C'_0:=C_0\cap(\delta\times c''\delta)$ is not separable from $\text{Gr}(c_{|\delta})$ by a pot$(\mathbf{\Pi}^0_1)$ set.

• By Lemma 7.5 applied to $\delta_0:=\delta$ and $\delta_1:=c''\delta$ there are $n<p$ and $y_0\in Y_0$ such that $(cc_n^{-1}c_p)(y_0)$ and $(cc_n^{-1}c)(y_0)$ are defined and different, which contradicts Lemma 7.7.(f). $\square$

Remark. We recover the algebraic relation "$g_n=g_n\circ g_p$ if $n<p$" that was already present in Section 3 of [L5] mentioned just after the statement of Theorem 7.1.

Theorem 7.8 *There is no tuple $\big((Z_i)_{i\in 2},S_0,S_1\big)$, where the Z_i's are Polish spaces and S_0, S_1 disjoint analytic subsets of $\Pi_{i\in 2}\ Z_i$, such that for any tuple $\big((X_i)_{i\in 2},B_0,B_1\big)$ of the same type exactly one of the following holds:*
(a) The set B_0 is separable from B_1 by a pot$(\mathbf{\Delta}^0_1)$ set.
(b) The inequality $\big((Z_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},B_0,B_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 7.1. This time, S_0 is not separable from S_1 by a pot$(\mathbf{\Delta}^0_1)$ set.

• Note that $A_1=\bigcup_{n\in\omega}\ \text{Gr}(H_n)$, where $H_n:N_{s_n0}\rightarrow N_{s_n1}$ is a partial homeomorphism with clopen domain and range. The crucial properties of $(s_n)_{n\in\omega}\subseteq 2^{<\omega}$ is that it is dense and $|s_n|=n$. We can easily ensure this in such a way that $(s_{2n})_{n\in\omega}$ and $(s_{2n+1})_{n\in\omega}$ are dense. We set $B_\varepsilon:=\bigcup_{n\in\omega}\ \text{Gr}(H_{2n+\varepsilon})$. The previous remark implies that $\Delta(2^\omega)=\overline{B_\varepsilon}\setminus B_\varepsilon$. By Lemma 7.6, B_0 is not separable from B_1 by a pot$(\mathbf{\Delta}^0_1)$ set. So here again $f_{i|\Pi''_iS_0}$ is countable-to-one for each $i\in 2$, and S_0, S_1 are locally countable by Lemma 7.3.

• Lemma 7.7 gives a tuple $\Big(\big(\bigcap_{n\in\omega}\ \mathrm{Gr}(c_n),2^\omega\big),C^0_0,C^1_0\Big)$. Note that $\Big(\big(\bigcap_{n\in\omega}\ \mathrm{Gr}(c_n),2^\omega\big),C^0_0,C^1_0\Big)$ satisfies condition (b) in Theorem 7.8.

• We change the topology on 2^ω into a finer Polish topology τ so that the sets $f_i''O^i_{n_i}$ become clopen and the maps $(f_{i|O^i_{n_i}})^{-1}$ become continuous. Now

$$\overline{D_0}^{\tau^2}\cap\overline{D_1}^{\tau^2}\subseteq\overline{B_0}\cap\overline{B_1}=\big(B_0\cup\Delta(2^\omega)\big)\cap\big(B_1\cup\Delta(2^\omega)\big)=\Delta(2^\omega).$$

So there is a Borel subset D of 2^ω such that $\overline{D_0}^{\tau^2}\cap\overline{D_1}^{\tau^2}=\Delta(D)$, and $D\subseteq\bigcap_{i\in 2}\ f_i''O^i_{n_i}$.

• Let us prove that $D_0\cap D^2$ is not separable from $D_1\cap D^2$ by a pot$(\mathbf{\Delta}^0_1)$ set.

We argue by contradiction, which gives $P\in\mathrm{pot}(\mathbf{\Delta}^0_1)$ such that $D_0\cap D^2\subseteq P\subseteq D^2\setminus D_1$. The sets $\overline{D_0}^{\tau^2}\cap(\neg D\times 2^\omega)$ and $\overline{D_1}^{\tau^2}\cap(\neg D\times 2^\omega)$ are disjoint, pot$(\mathbf{\Pi}^0_1)$, so that they are separable by Δ_l in pot$(\mathbf{\Delta}^0_1)$. Similarly, there is $\Delta_r\in\mathrm{pot}(\mathbf{\Delta}^0_1)$ which separates $\overline{D_0}^{\tau^2}\cap(2^\omega\times\neg D)$ from $\overline{D_1}^{\tau^2}\cap(2^\omega\times\neg D)$. Now

$$D_0\subseteq P\cup\big(D_0\cap(\neg D\times 2^\omega)\big)\cup\big(D_0\cap(2^\omega\times\neg D)\big)\subseteq P\cup\big(\Delta_l\cap(\neg D\times 2^\omega)\big)\cup\big(\Delta_r\cap(2^\omega\times\neg D)\big)\subseteq\neg D_1$$

which is absurd since $P\cup\big(\Delta_l\cap(\neg D\times 2^\omega)\big)\cup\big(\Delta_r\cap(2^\omega\times\neg D)\big)\in\mathrm{pot}(\mathbf{\Delta}^0_1)$.

• Let us prove that $D_0\cap D^2$ is not separable from $\Delta(D)$ by a pot$(\mathbf{\Pi}^0_1)$ set.

We argue by contradiction, which gives $Q\in\mathrm{pot}(\mathbf{\Pi}^0_1)$ such that $D_0\cap D^2\subseteq Q\subseteq D^2\setminus\Delta(D)$. The sets Q and $\Delta(D)$ are disjoint, pot$(\mathbf{\Pi}^0_1)$, so that there is R in pot$(\mathbf{\Delta}^0_1)$ such that $Q\subseteq R\subseteq D^2\setminus\Delta(D)$. The sets $\overline{D_0}^{\tau^2}\cap R$ and $\overline{D_1}^{\tau^2}\cap R$ are disjoint, pot$(\mathbf{\Pi}^0_1)$, so that there is S in pot$(\mathbf{\Delta}^0_1)$ such that $\overline{D_0}^{\tau^2}\cap R\subseteq S\subseteq R\setminus\overline{D_1}^{\tau^2}$. But S separates $D_0\cap D^2$ from $D_1\cap D^2$, which contradicts the previous point.

• Note that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]\subseteq\overline{C^0_0}\cap\overline{C^1_0}\cap(\bigcap_{n\in\omega}\ \mathrm{Dom}(c_n)\times 2^\omega)\subseteq\mathrm{Gr}(c)$. We conclude as in the proof of Theorem 7.1. $\square$

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